The heat equation

Many physical processes are governed by partial differential equations. One such phenomenon is the temperature of a rod. In this chapter, we will examine exactly that.

1 Deriving the heat equation

1.1 What is a partial differential equation?

In physical problems, many variables depend on multiple other variables. For example, the temperature $u(x,t)$ $[K]$ can depend on both position and time. Such variables don't have normal derivates like du/dt. Instead, they have **partial derivates**, like $\partial u/\partial x$ and $\partial u/\partial t$.

We can set up an equation with multiple partial derivatives. We would then get a **partial differential equation** (PDE). So a partial differential equation is an equation containing partial derivatives. If a differential equation does not contain partial derivatives, it's only an **ordinary differential equation** (ODE).

1.2 Conservation of energy for a one-dimensional rod

Let's consider a one-dimensional rod of length L. We define the **thermal energy density** $e(x,t)$ $[J/m^3]$ as the energy per unit volume. It depends not only on the position in the rod, but also on time. This is because it can change as time passes by.

There are two reasons why e can vary in time. First, there is the **heat flux** $\phi(x,t)$ $[J/m^2 s]$. This is the heat flowing to the right through a unit cross-section per unit time. Second, there can be internal heat creation due to **heat sources**. The amount of heat created is denoted by $Q(x,t)$ $[J/m^3 s]$.

We can now apply the law of conservation of energy to our rod. Let's examine a thin slice. To be more specific, we examine the rate of change of energy in it. This must be equal to the heat created, plus the heat flowing in, minus the heat flowing out. This gives us

$$\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q. \tag{1.1}$$

This equation is called the **integral conservation law**.

1.3 Deriving the heat equation for a one-dimensional rod

Let's define $u(x,t)$ $[K]$ as the temperature in the rod. There is a relation between this temperature u, and the variable e. But to find it, we first have to define two other variables.

Let's define the **mass density** $\rho(x)$ $[kg/m^3]$ as the mass per unit volume. (Usually ρ varies with temperature, and thus also with time. However, these variations are usually small. We thus neglect them.) We also define the **specific heat** $c(x)$ $[j/kg\,K]$ as the heat necessary to raise the temperature of a $1kg$-mass by 1 Kelvin. (We assume it to be constant in time for the same reasons as for the mass density.)

Using the above definitions, we can derive that

$$e(x,t) = c(x)\rho(x)u(x,t). \tag{1.2}$$

This transforms the integral conservation law into

$$c\rho\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + Q. \tag{1.3}$$

Usually Q is known. But u and ϕ are not. So we still have two unknowns. But you may be wondering, doesn't the heat flow depend on the temperature as well? In fact, it does. The heat flow depends on temperature differences. The higher these differences, the higher the heat flow. So we can say that

$$\phi(x,t) = -K_0 \frac{\partial u}{\partial x}. \tag{1.4}$$

This equation is called **Fourier's law of heat conduction**. The variable $K_0(x)$ $[J/Kms]$ is called the **thermal conductivity**. (It is also assumed to be constant for varying temperatures.) The above law now reduces our differential equation into

$$c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q. \tag{1.5}$$

If there are no heat sources (and thus $Q = 0$), we can rewrite this to

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \qquad \text{where} \qquad k = \frac{K_0}{c\rho}. \tag{1.6}$$

The important equation above is called the **heat equation**. By the way, k $[m^2/s]$ is called the **thermal diffusivity**.

1.4 Initial and boundary conditions

When solving a partial differential equation, we will need initial and boundary conditions. But what conditions do we exactly need?

If we look at the heat equation, we see that there is only a first time-derivative of u. So we need only one **initial condition** (IC). (An initial condition is a condition at $t = 0$.) Usually such a condition takes the form $u(x, 0) = f(x)$.

However, the heat equation contains a second derivative with respect to x. So we will need two **boundary conditions** (BC). (A boundary condition is a condition at a specified position.) These boundary conditions are usually the temperatures at the edges of the rod. So, $u(0,t) = T_1(t)$ and $u(L,t) = T_2(t)$.

However, it is also possible to set the heat flow ϕ (or equivalently $\partial u/\partial x$) at the edges of the rod. We would then have values given for $\partial u(0,t)/\partial x$ and $\partial u(L,t)/\partial x$. If the rod is perfectly insulated at its edges, then $\phi = 0$ and thus also $\partial u/\partial x = 0$ at the edges.

It is of course also possible to combine the two possibilities above. In that case, we deal with **Newton's law of cooling**. The heat flow then depends on the difference in temperature with respect to a certain reference temperature $u_B(t)$. This would give us

$$-K_0(0)\frac{\partial u}{\partial x}(0,t) = -H\left(u(0,t) - u_B(t)\right) \qquad \text{and} \qquad -K_0(L)\frac{\partial u}{\partial x}(L,t) = H\left(u(L,t) - u_B(t)\right). \tag{1.7}$$

Here H is the **heat transfer coefficient**. Note that if $H = 0$, we are dealing with an insulated edge. On the other hand, if we have $H = \infty$, we would have a constant temperature at the edge.

2 Special cases of the heat equation

2.1 Perfect thermal contact

Let's suppose we have two rods of length L. We can connect them to each other, such that their edges are in contact. So one rod goes from $x = 0$ to $x = L$, while the second goes from $x = L$ to $x = 2L$. We can make this connection in such a way that there is **perfect thermal contact**. You may wonder, what does that mean? Well, it means two things.

First of all the temperature u at the edges of both rods must be equal. We write this as $u(L-,t) = u(L+,t)$. In words, if we approach the point $x = L$ from the left (negative) side, we find the same temperature as if we approach it from the right (positive) side.

But, perfect thermal contact also means that no heat is lost. The energy that exits the first rod enters the second rod. In an equation this means

$$\phi(L-,t) = \phi(L+,t), \qquad \text{or, equivalently,} \qquad K_0(L-)\frac{\partial u}{\partial x}(L-,t) = K_0(L+)\frac{\partial u}{\partial x}(L+,t). \tag{2.1}$$

2.2 Finding the steady-state solution

Let's suppose we have a heat problem where $Q = 0$ and $u(x,0) = f(x)$. Also suppose that our boundary conditions are constant. (They don't change in time.) Then we can expect that, after a while, the temperature $u(x,t)$ will not change in time anymore. The corresponding solution for $u(x,t)$ is called the **equilibrium** or **steady-state solution**.

How can we find this solution? Well, we know that $\partial u/\partial t = 0$. So also $\partial^2 u/\partial x^2 = 0$. This means that the temperature is given by $u(x,t) = C_1 x + C_2$. Using the boundary conditions, we can often find C_1 and C_2. Only if both rods have an insulated edge, we can't find both constants yet. In that case we would have to find C_2 using the initial condition. This goes according to

$$C_2 = \frac{1}{L}\int_0^L f(x)\,dx. \tag{2.2}$$

2.3 The heat equation in 3D

What happens when we don't have a one-dimensional rod, but a three-dimensional body? In that case we can also derive a heat equation. There are some small differences though.

This time the temperature $u(x,y,z,t)$ depends on a lot more variables, as does the heat flow ϕ. Also, the heat flow has a direction, so it is a vector (written as ϕ). It thus has a divergence $\nabla \cdot \phi$. The relation between ϕ and u is now given by $\phi = -K_0 \nabla u$. Using this data, we can derive that the **3-dimensional heat equation** becomes

$$c\rho\frac{\partial u}{\partial t} = K_0 \nabla^2 u + Q. \tag{2.3}$$

Here ∇^2 is the **Laplacian operator**, defined as

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}. \tag{2.4}$$

Let's look at the conditions. The intial condition takes the simple form of $u(x,y,z,0) = f(x,y,z)$. However, the boundary conditions are slightly more difficult. We can set the temperature $u(x,y,z,t)$ at the edge of our body at a certain value. We could also set the heat flow at the edge of our body at a certain value. We would thus set $\nabla u \cdot \hat{\mathbf{n}}$. (Here, $\hat{\mathbf{n}}$ is the unit vector at the edge, pointing outward.) We also have a 3-dimensional version of **Newton's law of cooling**. This would be

$$-K_0 \nabla u \cdot \hat{\mathbf{n}} = H\left(u - u_B\right). \tag{2.5}$$

What would happen if we try to find the steady-state solution in 3D? In that case, we would have to solve the equation $\nabla^2 u = -Q/K_0$. This equation is called **Poisson's equation**. If we also have $Q = 0$, we would have to solve $\nabla^2 u = 0$. This equation is called **Laplace's equation**. We will solve this later in this chapter.

3 Basic concepts needed to solve the heat equation

It is almost time for us to solve the heat equation. However, before we do that, we will have to look at some other things first.

3.1 Linear operators and linear equations

A linear operator is some operator L for which

$$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2), \tag{3.1}$$

where c_1 and c_2 are constants. For example, the **heat operator**

$$\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2} \tag{3.2}$$

is a linear operator. A **linear equation** for u is an equation of the form $L(u) = f$, with the function f known. If $f = 0$, we have a **linear homogeneous equation**. Linear homogeneous equations have a certain advantage. We can apply the principle of superposition to it. Suppose we would have two solutions u_1 and u_2. Then also $c_1u_1 + c_2u_2$ is a solution, for any constants c_1 and c_2.

3.2 Orthogonality

The property of orthogonality comes in very handy when solving heat equations. So let's examine it. We say that two function $f(x)$ and $g(x)$ are **orthogonal** on the interval $[0, L]$ if

$$\int_0^L f(x)g(x)dx = 0. \tag{3.3}$$

It can be shown that the functions $\sin(n\pi x/L)$ and $\sin(m\pi x/L)$ (with n and m positive integers) are orthogonal if $n \neq m$. And the same goes for cosines. That comes in handy! In fact, the general rules for the interval $[0, L]$ are

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \begin{cases} 0 & \text{if } m \neq n, \\ L/2 & \text{if } m = n. \end{cases} \tag{3.4}$$

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \begin{cases} 0 & \text{if } m \neq n, \\ L/2 & \text{if } m = n \neq 0, \\ L & \text{if } m = n = 0. \end{cases} \tag{3.5}$$

Sometimes, however, we are examining the interval $[-L, L]$. In this case all values double. So,

$$\int_{-L}^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \begin{cases} 0 & \text{if } m \neq n, \\ L & \text{if } m = n. \end{cases} \tag{3.6}$$

$$\int_{-L}^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \begin{cases} 0 & \text{if } m \neq n, \\ L & \text{if } m = n \neq 0, \\ 2L & \text{if } m = n = 0. \end{cases} \tag{3.7}$$

By the way, the functions $\sin(n\pi x/L)$ and $\cos(m\pi x/L)$ are always orthogonal on the interval $[-L, L]$. So for every n and m we have

$$\int_{-L}^L \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = 0. \tag{3.8}$$

4 Solving method for the heat equation

In this part we will present a basic method to solve the heat equation.

4.1 Introducing the method of separation of variables

Let's try to solve the homogeneous heat equation

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0. \tag{4.1}$$

Of course, there are also an initial condition $u(x,0) = f(x)$ and two boundary conditions. To solve this problem, we use the **method of separation of variables**. According to this method, we assume that we can write $u(x,t)$ as $X(x)T(t)$. Here, the function $X(x)$ only depends on x and $T(t)$ only depends on t. We can now rewrite the above equation to

$$\frac{1}{kT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dX^2}. \tag{4.2}$$

We have reduced the PDE to an ODE! It can also be shown that both sides of the above equation equal a certain constant $-\lambda$. Here, λ is called the **separation constant**. We thus get two ordinary differential equations, being

$$\frac{d^2X}{dx^2} = -\lambda X \qquad \text{and} \qquad \frac{dT}{dt} = -\lambda kT. \tag{4.3}$$

Solving the latter one is easy. The solution is

$$T(t) = ce^{-\lambda kT}, \tag{4.4}$$

where c is a constant. (It depends on the initial conditions.) However, solving the equation for X is a bit more difficult. In fact, we will find that it can only be solved for certain values of λ. These values are called the **eigenvalues** of the equation. The corresponding solutions for $X(x)$ are the **eigenfunctions**. Let's take a look at how we can find them.

4.2 Finding the eigenvalues and the eigenfunctions

We want to solve the ODE

$$\frac{d^2X}{dx^2} = -\lambda X. \tag{4.5}$$

We see that $X = 0$ is a solution. We call this the **trivial solution**, in which we are not interested. If we ignore this solution, we can distinguish three cases:

- $\lambda < 0$. In this case the general solution for $X(x)$ is

  $$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}. \tag{4.6}$$

  When applying boundary conditions, we usually only find the trivial solution. Only in certain special cases will there be eigenvalues $\lambda < 0$.

- $\lambda = 0$. Now we would find the solution $X(x) = c_1 x + c_2$. Using the boundary conditions, we can solve for c_1 and c_2. Sometimes it turns out that $c_1 = c_2 = 0$. In this case $X(x)$ is the trivial solution, and $\lambda = 0$ is not an eigenvalue. Sometimes, however, $X(x)$ is not the trivial solution. In this case $\lambda = 0$ actually is an eigenvalue, with the corresponding eigenfunction $X(x)$.

- $\lambda > 0$. In this case the general solution for $X(x)$ is

$$X(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x. \tag{4.7}$$

We can apply the boundary conditions to the above equation. We should then look for a solution for which $X(x) \neq 0$. (We don't want the trivial solution.) It usually turns out that there are only solutions for certain λ. These λ are the eigenvalues. The corresponding solutions for $X(x)$ are the eigenfunctions.

The above method to find the eigenfunctions might seem a bit odd initially. However, they will become more clear after the examples that will be treated in the upcoming part.

4.3 Putting together the eigenfunctions

Now several eigenvalues λ_n and eigenfunctions $X_n(x)$ are known. We can see that every function $X_n(x)T_n(t)$ is a solution to the PDE. So, the general solution is then given by all linear combinations of these solutions. In an equation this becomes

$$u(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t). \tag{4.8}$$

To find the coefficients c_n, we must use the initial condition. Just insert $t = 0$ in the above equation. Then, using orthogonality, you can find expressions for the coefficients c_n. Although this is also slightly more difficult than it seems, we will demonstrate it using the following examples.

5 Example solutions of the heat equation

5.1 First example: Both edges having $u = 0$

Let's suppose our rod has both its sides kept at a constant temperature $u = 0$. So our boundary conditions are $u(0,t) = 0$ and $u(L,t) = 0$. From this follows that $X(0) = 0$ and $X(L) = 0$.

If $\lambda < 0$, then we can show that $X(x) = 0$ as well. So we only find the trivial solution.

What happens if $\lambda = 0$? In this case $X(x) = c_1 x + c_2$. Applying the boundary conditions will give $c_1 = c_2 = 0$. So, $\lambda = 0$ is not an eigenvalue.

However, if $\lambda > 0$, we will find some eigenvalues. We can apply the boundary condition $X(0) = 0$ in equation 4.7. We will then find $c_1 = 0$. If we also apply $X(L) = 0$, we will find $c_2 \sin \sqrt{\lambda} L = 0$. If we also have $c_2 = 0$, we would only find the trivial solution. So, instead, we must have $\sin \sqrt{\lambda} L = 0$. This can only be true if $\sqrt{\lambda} L = n\pi$, with $n = 1, 2, 3, \ldots$. So our eigenvalues λ_n and eigenfunctions $X_n(x)$ are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \text{and} \qquad X_n(x) = \sin\left(\frac{n\pi x}{l}\right). \tag{5.1}$$

Note that we have dropped the constant at $X_n(x)$. We are allowed to do this because constants don't really matter with eigenfunctions. If $X_n(x)$ is an eigenfunction, then so is any multiple of it.

So, we can now see that any function of the form

$$u_n(x,t) = X(x)T(t) = \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda k t}, \qquad \text{with } n = 1, 2, 3, \ldots \tag{5.2}$$

is a solution satisfying the differential equation. In fact, any linear combination of the above solutions is a solution. So we could say that our general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda k t}. \tag{5.3}$$

However, there is one condition we haven't satisfied yet: the initial condition. And we can use this condition to find the constants B_n. To satisfy the initial condition, we must have

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right). \tag{5.4}$$

Now we must apply the property of orthogonality to find the coefficients B_n. To do this, we multiply by $\sin(m\pi x/L)$ and integrate from 0 to L. We then get

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right) = \sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right). \tag{5.5}$$

Now we can note that every term in the sum on the right drops out (is zero), except for the term with $n = m$. The right side of the equation thus reduces to $B_m L/2$. It follows that

$$B_m = \frac{2}{L}\int_0^L f(x)\sin\frac{m\pi x}{L}dx. \tag{5.6}$$

Using this equation, we can find our constants B_n, and thus also our unique solution for $u(x,t)$.

5.2 Second example: Both edges being insulated

In the second example we examine a rod with both its edges insulated. So our boundary conditions are $\partial u/\partial x(0,t) = 0$ and $\partial u/\partial x(L,t) = 0$.

If $\lambda < 0$, then we can again show that $X(x) = 0$ as well. So we only find the trivial solution.

Let's examine the case where $\lambda = 0$. We now do find a non-trivial solution. Once more we have $X_0(x) = c_1 x + c_2$. Both boundary conditions imply that $c_1 = 0$. However, c_2 can be anything. So we have found a non-trivial solution. Thus $\lambda = 0$ is an eigenvalue! The corresponding eigenfunction is $X_0(x) = 1$. (Remember that we were allowed to ignore constants when examining eigenfunctions.)

Now let's consider the case $\lambda > 0$. This time our solutions will be

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \text{and} \qquad X_n(x) = \cos\left(\frac{n\pi x}{l}\right). \tag{5.7}$$

The resulting general solution of our PDE (before applying the initial conditions) will then be

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) e^{-\lambda k t}. \tag{5.8}$$

The constants A_0 and A_n follow from our initial condition. This time we must multiply by $\cos(n\pi x/L)$ and integrate from 0 to L. We then find that

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx \qquad \text{and} \qquad A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx. \tag{5.9}$$

5.3 Laplace's equation

We can also extend the method of separation of variables to a two-dimensional plate with width L and height H. However, things get more difficult now. So to make it easier, we only want to find the steady-state solution (with $\partial u/\partial t = 0$). This turns our differential equation into Laplace's equation, being

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{5.10}$$

To demonstrate the solution method, we will show one example. Let's assume that we have boundary conditions $u(0,y) = 0$, $u(L,y) = f(y)$, $u(x,0) = 0$ and $u(x,H) = 0$. To apply the method of separation of variables, we assume that $u(x,y) = X(x)Y(y)$. Our differential equation now turns into

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda. \tag{5.11}$$

So we again have two ODEs. For $Y(y)$, we have two rather easy boundary conditions, being $Y(0) = 0$ and $Y(H) = 0$. So let's focus on the y-part. We can solve this using methods we have seen earlier. We find as eigenvalues and eigenfunctions

$$\lambda_n = (n\pi/H)^2 \qquad \text{and} \qquad Y_n(y) = \sin\frac{n\pi y}{H}. \tag{5.12}$$

Now let's find $X(x)$ as well. we now have to solve

$$\frac{d^2X}{dx^2} = \lambda X, \qquad \text{or equivalently} \qquad \frac{d^2X}{dx^2} = \left(\frac{n\pi}{H}\right)^2 X. \tag{5.13}$$

The solution for this equation is

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x} \qquad \text{or equivalently} \qquad X(x) = c_1\cosh\sqrt{\lambda}x + c_2\sinh\sqrt{\lambda}x. \tag{5.14}$$

We can use either of the above relations. However, in this case the relation with sinh and cosh is more convenient, since then one term will drop out. So let's use that one.

One of our boundary conditions is $X(0) = 0$. If we apply this, we find that $c_1 = 0$ and the part with cosh will disappear. We thus have as eigenfunctions $X_n(x) = \sinh\sqrt{\lambda}x$. Our general solution for u then becomes

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi y}{H}\sinh\frac{n\pi x}{H}. \tag{5.15}$$

However, we haven't applied one boundary condition yet, being $u(L,y) = f(y)$. We can use this to find the constants c_n. Inserting $x = L$ in the above equation gives

$$f(y) = u(L,y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi y}{H}\sinh\frac{n\pi L}{H}. \tag{5.16}$$

It can then be derived, using the porperty of orthogonality, that

$$c_n = \frac{2}{H\sinh\frac{n\pi L}{H}}\int_0^H f(y)\sin\frac{n\pi y}{H}dy. \tag{5.17}$$

And this completes our solution to the problem.