1.3.2 Algebra and Calculus Tools

This section provides some definitions and elementary properties about logarithms, probability, permutations, summation formulas, and common mathematical sequences and series. (A series is the sum of a sequence in this context.) We will introduce additional mathematical tools for recurrence equations in Chapter 3. You can find formulas not derived here by consulting the sources in Notes and References at the end of the chapter.

Floor and Ceiling Functions

For any real number \( x \), \([x]\) (read “floor of \( x \)”) is the largest integer less than or equal to \( x \). \([x]\) (read “ceiling of \( x \)”) is the smallest integer greater than or equal to \( x \). For example, \([2.9] = 2\), and \([6.1] = 7\).

Logarithms

The logarithm function, usually to the base 2, is the mathematical tool used most extensively in this book. Although logarithms do not occur very frequently in natural sciences, they are prevalent in computer science.

Definition 1.3 Logarithm function and logarithmic base

For \( b > 1 \) and \( x > 0 \), \( \log_b x \) (read “log to the base \( b \) of \( x \)”) is that real number \( L \) such that \( b^L = x \); that is, \( \log_b x \) is the power to which \( b \) must be raised to get \( x \).

The following properties of logarithms follow easily from the definition.

Lemma 1.1 Let \( x \) and \( y \) be arbitrary positive real numbers, let \( a \) be any real number, and let \( b > 1 \) and \( c > 1 \) be real numbers.

1. \( \log_b x \) is a strictly increasing function, that is, if \( x > y \), then \( \log_b x > \log_b y \).
2. \( \log_b x \) is a one-to-one function, that is, if \( \log_b x = \log_b y \), then \( x = y \).
3. \( \log_b 1 = 0 \).
4. \( \log_b b^a = a \).
5. \( \log_b (xy) = \log_b x + \log_b y \).
6. \( \log_b (x^a) = a \log_b x \).
7. \( \log_b x = \frac{\log_a x}{\log_a b} \).
8. To convert from one base to another: \( \log_a x = \frac{\log_b x}{\log_b a} \).

Since the log to the base 2 is used most often in computational complexity, there is a special notation for it: “lg”; that is, \( \log_2 x \). The natural logarithm (log to the base \( e \)) is denoted by “ln”; that is, \( \ln x = \log_e x \). When \( \log(x) \) is used without any base being mentioned, it means the statement is true for any base.

Sometimes the logarithm function is applied to itself. The notation \( \log \log(x) \) means \( \log(\log(x)) \). The notation \( \log^{[p]}(x) \) means \( p \) applications, so \( \log^{[2]}(x) \) is the same as \( \log \log(x) \).