Confidence Intervals for Nonlinear Regression Parameters

1. The matrix $\boldsymbol{F}(\boldsymbol{\theta}) = \boldsymbol{F}$ plays the role in nonlinear least squares the $\boldsymbol{X}$ matrix plays in linear least squares.

2. If $\boldsymbol{\varepsilon} \sim N(\mathbf{0}, \boldsymbol{I}\sigma^2)$, then $\hat{\boldsymbol{\theta}}$ is *approximately* normally distributed with mean $\boldsymbol{\theta}$ and $Var(\hat{\boldsymbol{\theta}}) = (\boldsymbol{F'F})^{-1}\sigma^2$. (Gallant 1987)

3. The residual sum of squares $S(\hat{\boldsymbol{\theta}})$ when divided by σ^2, has *approximately* a chi-squared distribution with $n-p$ degrees of freedom.

4. In practice, $\boldsymbol{F}(\boldsymbol{\theta})$ is computed as $\boldsymbol{F}(\hat{\boldsymbol{\theta}}) = \hat{\boldsymbol{F}}$ and σ^2 is estimated with $s^2 = S(\hat{\boldsymbol{\theta}})/(n-p)$, so that the estimated variance-covariance matrix for $\hat{\boldsymbol{\theta}}$ is $\boldsymbol{s}^2(\hat{\boldsymbol{\theta}}) = (\hat{\boldsymbol{F}}'\hat{\boldsymbol{F}})^{-1}s^2$

5. Confidence Intervals for $\boldsymbol{\theta}$. Let $C = \boldsymbol{K'\theta}$ be any linear function of interest (point estimate is $\hat{C} = \boldsymbol{K'}\hat{\boldsymbol{\theta}}$).

   $100(1-\alpha)\%$ CI estimate of C is: $\hat{C} \pm t_{\alpha/2}(n-p)s(\hat{C})$

   where $s(\hat{C}) = \{\boldsymbol{K'}[s^2(\hat{\boldsymbol{\theta}})]\boldsymbol{K}\}^{1/2}$

6. Likelihood Ratio Theory: Approximate joint $100(1-\alpha)\%$ confidence region for $\hat{\boldsymbol{\theta}}$: Set of $\hat{\boldsymbol{\theta}}$ s.t.
$$S(\boldsymbol{\theta}) \leq S(\hat{\boldsymbol{\theta}}) \left[1 + \frac{p}{n-p}\mathrm{F}_{\alpha}(p, n-p)\right]$$