STAT 700
Homework 1 Problems
due Wed. Sept. 12

2 Problems. Show all work.
This is an group assignment and should be **handwritten**.

1. Let $X_1, X_2, \ldots, X_n$ be independent normal random variables with means μ_i and variances σ_i^2. Let $Y = \sum_{i=1}^n \alpha_i X_i$ where α_i are constants. Use moment generating functions to show that Y is normally distributed and find its mean and variance.

Recall, the moment generating function (mgf) of a normal random variable X with mean μ and variance σ^2 is

$$M_X(t) = e^{\mu t + \sigma^2 t^2/2}.$$

2. Consider the linear model from class,

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}.$$

Assume that the ε_i are independent $N(0, \sigma^2)$ random variables or equivalently

$$\boldsymbol{Y} \sim N_n(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2 \boldsymbol{I}_n).$$

Also, assume that $\boldsymbol{X}'\boldsymbol{X}$ is invertible.

Suppose we fit the model,

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \tag{1}$$

where $\boldsymbol{X}$ involves $r = 3$ independent variables. Assume that there are $n = 10$ observations and ε_i are independent $N(0, \sigma^2)$ random variables.

(a) Write model (1) in full matrix notation, indicating all individual elements and dimensions of matrices. You may assume there is an intercept.
(b) Give the form of the least squares estimator of $\boldsymbol{\beta}$, for the fit to model (1). Call it $\hat{\boldsymbol{\beta}}$. You may use the notation of model (1).
(c) Find $E(\hat{\boldsymbol{\beta}})$ for model (1). Is $\hat{\boldsymbol{\beta}}$ a biased or unbiased estimator of $\boldsymbol{\beta}$? Explain. Find the distribution of $\hat{\boldsymbol{\beta}}$.

Now assume that the true model involves an additional $s = 2$ independent variables contained in $\boldsymbol{W}$, so the true model is

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{W}\boldsymbol{\gamma} + \boldsymbol{\varepsilon} \tag{2}$$

where $\boldsymbol{\gamma}$ is the vector of regression coefficients for the independent variables contained in $\boldsymbol{W}$.

(d) Write model (2) in full matrix notation, indicating all individual elements and dimensions of matrices. To save us from re-writing part (a), you can just write down the matrix $\boldsymbol{W}$ and the vector $\boldsymbol{\gamma}$. Note: matrix $\boldsymbol{W}$ should not have a column of 1's.

(e) Under the true model (2), find $E(\hat{\boldsymbol{\beta}})$. In general, is $\hat{\boldsymbol{\beta}}$ an biased or unbiased estimator of $\boldsymbol{\beta}$? Explain. Find the distribution of $\hat{\boldsymbol{\beta}}$.