6.2 χ^2, t, F Distribution (and gamma, beta)

Normal Distribution

Consider the integral

$$I = \int_{-\infty}^{\infty} e^{-y^2/2} dy$$

To evaluate the intgral, note that $I > 0$ and

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2 + z^2}{2}\right) dydz$$

This integral can be easily evaluated by changing to polar coordinates. $y = rsin(\theta)$ and $z = rcos(\theta)$. Then

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2} r dr d\theta$$

$$= \int_0^{2\pi} \left[-e^{-r^2/2} |_0^{\infty} \right] d\theta$$

$$= \int_0^{2\pi} d\theta = 2\pi$$

This implies that $I = \sqrt{2\pi}$ and

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = 1$$

If we introduce a new variable of integration

$$y = \frac{x - a}{b}$$

where $b > 0$, the integral becomes

$$\int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} \exp\left[\frac{-(x-a)^2}{2b^2}\right] dx = 1$$

This implies that

$$f(x) = \frac{1}{b\sqrt{2\pi}} \exp\left[\frac{-(x-a)^2}{2b^2}\right]$$

for $x \in (-\infty, \infty)$ satisfies the conditions of being a pdf. A random variable of the continuous type with a pdf of this form is said to have a **normal distribution**.

Let's find the mgf of a normal distribution.

$$M(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{b\sqrt{2\pi}} \exp\left[\frac{-(x-a)^2}{2b^2}\right] dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} \exp\left(-\frac{-2b^2tx + x^2 - 2ax + a^2}{2b^2}\right) dx$$

$$= \exp\left[-\frac{a^2 - (a+b^2t)^2}{2b^2}\right] \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} \exp\left[-\frac{(x-a-b^2t)^2}{2b^2}\right] dx$$

$$= \exp\left(at + \frac{b^2t^2}{2}\right)$$

Note that the exponential form of the mgf allows for simple derivatives

$$M'(t) = M(t)(a + b^2t)$$

and

$$M''(t) = M(t)(a + b^2 t)^2 + b^2 M(t)$$

$$\mu = M'(0) = a$$

$$\sigma^2 = M''(0) - \mu^2 = a^2 + b^2 - a^2 = b^2$$

Using these facts, we write the pdf of the normal distribution in its usual form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

for $x \in (-\infty, \infty)$. Also, we write the mgf as

$$M(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

Theorem If the random variable X is $N(\mu, \sigma^2)$, $\sigma^2 > 0$, then the random variable $W = (X - \mu)/\sigma$ is $N(0, 1)$.
Proof:

$$F(w) = P[\frac{X - \mu}{\sigma} \leq w] = P[X \leq w\sigma + \mu]$$

$$= \int_{-\infty}^{w\sigma+\mu} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx.$$

If we change variables letting $y = (x - \mu)/\sigma$ we have

$$F(w) = \int_{-\infty}^{w} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

Thus, the pdf $f(w) = F'(w)$ is just

$$f(w) = \frac{1}{\sqrt{2\pi}} e^{-w^2/2}$$

for $-\infty < w < \infty$, which shows that W is $N(0, 1)$.

Recall, the **gamma function** is defined by

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$$

for $\alpha > 0$.

If $\alpha = 1$,

$$\Gamma(1) = \int_0^\infty e^{-y} dy = 1$$

If $\alpha > 1$, integration by parts can be used to show that

$$\Gamma(a) = (\alpha - 1) \int_0^\infty y^{\alpha-2} e^{-y} dy = (\alpha - 1)\Gamma(\alpha - 1)$$

By iterating this, we see that when α is a positive integer $\Gamma(\alpha) = (\alpha - 1)!$.

In the integral defining $\Gamma(\alpha)$ let's have a change of variables $y = x/\beta$ for some $\beta > 0$. Then

$$\Gamma(\alpha) = \int_0^\infty \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-x/\beta} \left(\frac{1}{\beta}\right) dx$$

Then, we see that

$$1 = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx$$

When $\alpha > 0$, $\beta > 0$ we have

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$$

is a pdf for a continuous random variable with space $(0, \infty)$. A random variable with a pdf of this form is said to have a **gamma distribution** with parameters α and β.

Recall, we can find the mgf of a gamma distribution.

$$M(t) = \int_0^\infty \frac{e^{tx}}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx$$

Set $y = x(1-\beta t)/\beta$ for $t < 1/\beta$. Then

$$M(t) = \int_0^\infty \frac{\beta/(1-\beta t)}{\Gamma(\alpha)\beta^\alpha} \left(\frac{\beta y}{1-\beta t}\right)^{\alpha-1} e^{-y} dy$$

$$= \left(\frac{1}{1-\beta t}\right)^\alpha \int_0^\infty \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} dy$$

$$= \frac{1}{(1-\beta t)^\alpha}$$

for $t < \frac{1}{\beta}$.

$$M'(t) = \alpha\beta(1-\beta t)^{-\alpha-1}$$

$$M''(t) = \alpha(\alpha+1)\beta^2(1-\beta t)^{-\alpha-2}$$

So, we can find the mean and variance by

$$\mu = M'(0) = \alpha\beta$$

and

$$\sigma^2 = M''(0) - \mu^2 = \alpha\beta^2$$

An important special case is when $\alpha = r/2$ where r is a positive integer, and $\beta = 2$. A random variable X with pdf

$$f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}$$

for $x > 0$ is said to have a **chi-square distribution** with r **degrees of freedom**. The mgf for this distribution is

$$M(t) = (1 - 2t)^{-r/2}$$

for $t < 1/2$.

Example: Let X have the pdf

$$f(x) = 1$$

for $0 < x < 1$. Let $Y = -2ln(X)$. Then $x = g^{-1}(y) = e^{-y/2}$.

The space $\mathcal{A}$ is $\{x : 0 < x < 1\}$, which the one-to-one transformation $y = -2ln(x)$ maps onto $\mathcal{B}$.

$\mathcal{B}$= $\{y : 0 < y < \infty\}$.

The Jacobian of the transformation is

$$J = -\frac{1}{2}e^{-y/2}$$

Accordingly, the pdf of Y is

$$f(y) = f(e^{-y/2})|J| = \frac{1}{2}e^{-y/2}$$

for $0 < y < \infty$.

Recall the pdf of a chi-square distribution with r degress of freedom.

$$f(x) = \frac{1}{\Gamma(r/2)2^{r/2}}x^{r/2-1}e^{-x/2}$$

From this we see that $f(x) = f(y)$ when $r = 2$.

Definition (Book) If Z is a standard normal random variable, the distribution of $U = Z^2$ is called a chi-square distribution with 1 degree of freedom.

Theorem If the random variable X is $N(\mu, \sigma^2)$, then the random variable $V = (X - \mu)^2/\sigma^2$ is $\chi^2(1)$.

Beta Distribution

Let X_1 and X_2 be independent gamma variables with joint pdf

$$h(x_1, x_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x_1^{\alpha-1} x_2^{\beta-1} e^{-x_1-x_2}$$

for $0 < x_1 < \infty$ and $0 < x_2 < \infty$, where $\alpha > 0$, $\beta > 0$.

Let $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1+X_2}$.

$$y_1 = g_1(x_1, x_2) = x_1 + x_2$$

$$y_2 = g_2(x_1, x_2) = \frac{x_1}{x_1 + x_2}$$

$$x_1 = h_1(y_1, y_2) = y_1 y_2$$

$$x_2 = h_2(y_1, y_2) = y_1(1 - y_2)$$

$$J = \begin{vmatrix} y_2 & y_1 \\ (1-y_2) & -y_1 \end{vmatrix} = -y_1$$

The transformation is one-to-one and maps $\mathcal{A}$, the first quadrant of the x_1x_2 plane onto
$\mathcal{B}$=$\{(y_1, y_2) : 0 < y_1 < \infty, 0 < y_2 < 1\}$.

The joint pdf of Y_1, Y_2 is

$$\begin{aligned} f(y_1, y_2) &= \frac{y_1}{\Gamma(\alpha)\Gamma(\beta)}(y_1y_2)^{\alpha-1}[y_1(1-y_2)]^{\beta-1}e^{-y_1} \\ &= \frac{y_2^{\alpha-1}(1-y_2)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}y_1^{\alpha+\beta-1}e^{-y_1} \end{aligned}$$

for $(y_1, y_2) \in \mathcal{B}$.

Because $\mathcal{B}$ is a rectangular region and because $g(y_1, y_2)$ can be factored into a function of y_1 and a function of y_2, it follows that Y_1 and Y_2 are statistically independent.

The marginal pdf of Y_2 is

$$f_{Y_2}(y_2) = \frac{y_2^{\alpha-1}(1-y_2)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty y_1^{\alpha+\beta-1} e^{-y_1} dy_1$$
$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y_2^{\alpha-1}(1-y_2)^{\beta-1}$$

for $0 < y_2 < 1$.

This is the pdf of a **beta distribution** with parameters α and β.
Also, since $f(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2)$ we see that

$$f_{Y_1}(y_1) = \frac{1}{\Gamma(\alpha+\beta)} y_1^{\alpha+\beta-1} e^{-y_1}$$

for $0 < y_1 < \infty$.

Thus, we see that Y_1 has a gamma distribution with parameter values $\alpha+\beta$ and 1.

To find the mean and variance of the beta distribution, it is helpful to notice that from the pdf, it is clear that for all $\alpha > 0$ and $\beta > 0$,

$$\int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

The expected value of a random variable with a beta distribution is

$$\int_0^1 yg(y)dy = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 y^{\alpha}(1-y)^{\beta-1}dy$$

$$= \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+1+\beta)} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$= \frac{\alpha}{\alpha+\beta}$$

This follows from applying the fact that

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

To find the variance, we apply the same idea to find $E[Y^2]$ and use the fact that $var(Y) = E[Y^2] - \mu^2$.

$$\sigma^2 = \frac{\alpha\beta}{(\alpha + \beta + 1)(\alpha + \beta)^2}$$

t distribution

Let W and V be independent random variables for which W is $N(0,1)$ and V is $\chi^2(r)$.

$$f(w,v) = \frac{1}{\sqrt{2\pi}} e^{-w^2/2} \frac{1}{\Gamma(r/2)2^{r/2}} v^{r/2-1} e^{-r/2}$$

for $-\infty < w < \infty$, $0 < v < \infty$.

Define a new random variable T by

$$T = \frac{W}{\sqrt{V/r}}$$

To find the pdf $f_T(t)$ we use the change of variables technique with transformations

$t = \frac{w}{\sqrt{v/r}}$ and $u = v$.

These define a one-to-one transformation that maps

$\mathcal{A}$=$\{(w, v) : -\infty < w < \infty, 0 < v < \infty\}$ to

$\mathcal{B}$=$\{(t, u) : -\infty < t < \infty, 0 < u < \infty\}$.

The inverse transformations are

$w = \frac{t\sqrt{u}}{\sqrt{r}}$ and $v = u$.

Thus, it is easy to see that

$$|J| = \sqrt{u}/\sqrt{r}$$

By applying the change of variable technique, we see that the joint pdf of T and U is

$$f_{TU}(t,u) = f_{WV}(\frac{t\sqrt{u}}{\sqrt{r}}, u)|J|$$

$$= \frac{u^{r/2-1}}{\sqrt{2\pi}\Gamma(r/2)2^{r/2}} \exp\left[-\frac{u}{2}(1+t^2/r)\right] \frac{\sqrt{u}}{\sqrt{r}}$$

for $-\infty < t < \infty$, $0 < u < \infty$.

To find the marginal pdf of T we compute

$$f_T(t) = \int_0^\infty f(t,u)du$$

$$= \int_0^\infty \frac{u^{(r+1)/2-1}}{\sqrt{2\pi r}\Gamma(r/2)2^{r/2}} \exp\left[-\frac{u}{2}(1+t^2/r)\right] du$$

This simplifies with a change of variables $z = u[1+(t^2/r)]/2$.

$$f_T(t) = \int_0^\infty \frac{1}{\sqrt{2\pi r}\Gamma(r/2)2^{r/2}} \left(\frac{2z}{1+t^2/r}\right)^{(r+1)/2-1} e^{-z} \left(\frac{2}{1+t^2/r}\right) dz$$

$$= \frac{\Gamma[(r+1)/2]}{\sqrt{\pi r}\Gamma(r/2)(1+t^2/2)^{(r+1)/2}}$$

for $-\infty < t < \infty$.

A random variable with this pdf is said to have a **t distribution** with r **degrees of freedom**.

F Distribution

Let U and V be independent chi-square random variables with r_1 and r_2 degrees of freedom, respectively.

$$f(u,v) = \frac{u^{r_1/2-1} v^{r_2/2-1} e^{-(u+v)/2}}{\Gamma(r_1/2)\Gamma(r_2/2)2^{(r_1+r_2)/2}}$$

Define a new random variable

$$W = \frac{U/r_1}{V/r_2}$$

To find $f_W(w)$ we consider the transformation

$w = \frac{u/r_1}{v/r_2}$ and $z = v$.

This maps
$\mathcal{A}=\{(u,v) : 0 < u < \infty, 0 < v < \infty\}$ to
$\mathcal{B}=\{(w,z) : 0 < w < \infty, 0 < z < \infty\}$.

The inverse transformations are

$$u = (r_1/r_2)zw \text{ and } v = z.$$

This results in

$$|J| = (r_1/r_2)z$$

The joint pdf of W and Z by the change of variables technique is

$$f(w,z) = \frac{\left(\frac{r_1 zw}{r_2}\right)^{r_1/2-1} z^{r_2/2-1}}{\Gamma(r_1/2)\Gamma(r_2/2)2^{(r_1+r_2)/2}} \exp\left[-\frac{z}{2}\left(\frac{r_1 w}{r_2}+1\right)\right] \frac{r_1 z}{r_2}$$

for $(w,z) \in \mathcal{B}$.

The marginal pdf of W is

$$f_W(w) = \int_0^\infty f(w,z)dz$$

$$= \int_0^\infty \frac{(r_1/r_2)^{r_1/2}(w)^{r_1/2-1} z^{r_1+r_2/2-1}}{\Gamma(r_1/2)\Gamma(r_2/2)2^{(r_1+r_2)/2}} \exp\left[-\frac{z}{2}\left(\frac{r_1 w}{r_2}+1\right)\right] dz$$

We simplify this by changing the variable of integration to

$$y = \frac{z}{2}\left(\frac{r_1 w}{r_2}+1\right)$$

Then the pdf $f_W(w)$ is

$$\int_0^\infty \frac{(r_1/r_2)^{r_1/2}(w)^{r_1/2-1}}{\Gamma(r_1/2)\Gamma(r_2/2)2^{(r_1+r_2)/2}} \left(\frac{2y}{r_1 w/r_2+1}\right)^{(r_1+r_2)/2-1} e^{-y} \left(\frac{2}{r_1 w/r_2+1}\right) dy$$

$$= \frac{\Gamma[(r_1+r_2)/2](r_1/r_2)^{r_1/2}(w)^{r_1/2-1}}{\Gamma(r_1/2)\Gamma(r_2/2)(1+r_1 w/r_2)^{(r_1+r_2)/2}}$$

for $0 < w < \infty$.

A random variable with a pdf of this form is said to have an **F-distribution** with **numerator degrees of freedom** r_1 and **denominator degrees of freedom** r_2.