6.2 $\chi^2$, $t$, $F$ Distribution (and gamma, beta)

**Normal Distribution**

Consider the integral

$$ I = \int_{-\infty}^{\infty} e^{-y^2/2} dy $$

To evaluate the integral, note that $I > 0$ and

$$ I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2 + z^2}{2}\right) dydz $$

This integral can be easily evaluated by changing to polar coordinates. $y = r\sin(\theta)$ and $z = r\cos(\theta)$. Then
\[ I^2 = \int_0^{2\pi} \int_0^\infty e^{-r^2/2} r dr d\theta \]

\[ = \int_0^{2\pi} \left[ -e^{-r^2/2} \right]_0^\infty d\theta \]

\[ = \int_0^{2\pi} d\theta = 2\pi \]

This implies that \( I = \sqrt{2\pi} \) and

\[ \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = 1 \]
If we introduce a new variable of integration

\[ y = \frac{x - a}{b} \]

where \( b > 0 \), the integral becomes

\[ \int_{-\infty}^{\infty} \frac{1}{b \sqrt{2\pi}} \exp \left[ \frac{-(x - a)^2}{2b^2} \right] dx = 1 \]

This implies that

\[ f(x) = \frac{1}{b \sqrt{2\pi}} \exp \left[ \frac{-(x - a)^2}{2b^2} \right] \]

for \( x \in (-\infty, \infty) \) satisfies the conditions of being a pdf. A random variable of the continuous type with a pdf of this form is said to have a normal distribution.
Let’s find the mgf of a normal distribution.

\[
M(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{b\sqrt{2\pi}} \exp \left[ \frac{-(x-a)^2}{2b^2} \right] \, dx
\]

\[
= \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} \exp \left( -\frac{2b^2tx + x^2 - 2ax + a^2}{2b^2} \right) \, dx
\]

\[
= \exp \left[ -\frac{a^2 - (a + b^2t)^2}{2b^2} \right] \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} \exp \left[ -\frac{(x-a-b^2t)^2}{2b^2} \right] \, dx
\]

\[
= \exp \left( at + \frac{b^2t^2}{2} \right)
\]

Note that the exponential form of the mgf allows for simple derivatives

\[
M'(t) = M(t)(a + b^2t)
\]
and

\[ M''(t) = M(t)(a + b^2t)^2 + b^2M(t) \]

\[ \mu = M'(0) = a \]

\[ \sigma^2 = M''(0) - \mu^2 = a^2 + b^2 - a^2 = b^2 \]

Using these facts, we write the pdf of the normal distribution in its usual form

\[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{(x - \mu)^2}{2\sigma^2} \right] \]

for \( x \in (-\infty, \infty) \). Also, we write the mgf as

\[ M(t) = \exp \left( \mu t + \frac{\sigma^2 t^2}{2} \right) \]
**Theorem** If the random variable $X$ is $N(\mu, \sigma^2)$, $\sigma^2 > 0$, then the random variable $W = (X - \mu)/\sigma$ is $N(0, 1)$.

Proof:

$$F(w) = P\left[\frac{X - \mu}{\sigma} \leq w\right] = P[X \leq w\sigma + \mu]$$

$$= \int_{-\infty}^{w\sigma + \mu} \frac{1}{\sigma \sqrt{2\pi}} \exp\left[-\frac{(x - \mu)^2}{2\sigma^2}\right] dx.$$

If we change variables letting $y = (x - \mu)/\sigma$ we have

$$F(w) = \int_{-\infty}^{w} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

Thus, the pdf $f(w) = F'(w)$ is just

$$f(w) = \frac{1}{\sqrt{2\pi}} e^{-w^2/2}$$

for $-\infty < w < \infty$, which shows that $W$ is $N(0, 1)$. 
Recall, the **gamma function** is defined by

\[
\Gamma(\alpha) = \int_{0}^{\infty} y^{\alpha-1} e^{-y} dy
\]

for \( \alpha > 0 \).

If \( \alpha = 1 \),

\[
\Gamma(1) = \int_{0}^{\infty} e^{-y} dy = 1
\]

If \( \alpha > 1 \), integration by parts can be used to show that

\[
\Gamma(\alpha) = (\alpha - 1) \int_{0}^{\infty} y^{\alpha-2} e^{-y} dy = (\alpha - 1)\Gamma(\alpha - 1)
\]

By iterating this, we see that when \( \alpha \) is a positive integer \( \Gamma(\alpha) = (\alpha - 1)! \).
In the integral defining $\Gamma(\alpha)$ let’s have a change of variables $y = x/\beta$ for some $\beta > 0$. Then

$$\Gamma(\alpha) = \int_0^\infty \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-x/\beta} \left(\frac{1}{\beta}\right) dx$$

Then, we see that

$$1 = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx$$

When $\alpha > 0$, $\beta > 0$ we have

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$$

is a pdf for a continuous random variable with space $(0, \infty)$. A random variable with a pdf of this form is said to have a **gamma distribution** with parameters $\alpha$ and $\beta$. 
Recall, we can find the mgf of a gamma distribution.

\[ M(t) = \int_0^\infty \frac{e^{tx}}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \, dx \]

Set \( y = x(1 - \beta t)/\beta \) for \( t < 1/\beta \). Then

\[ M(t) = \int_0^\infty \frac{\beta/(1 - \beta t)}{\Gamma(\alpha)\beta^\alpha} \left( \frac{\beta y}{1 - \beta t} \right)^{\alpha-1} e^{-y} \, dy \]

\[ = \left( \frac{1}{1 - \beta t} \right)^\alpha \int_0^\infty \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} \, dy \]

\[ = \frac{1}{(1 - \beta t)^\alpha} \]

for \( t < \frac{1}{\beta} \).
\[ M'(t) = \alpha \beta (1 - \beta t)^{-\alpha - 1} \]

\[ M''(t) = \alpha (\alpha + 1) \beta^2 (1 - \beta t)^{-\alpha - 2} \]

So, we can find the mean and variance by

\[ \mu = M'(0) = \alpha \beta \]

and

\[ \sigma^2 = M''(0) - \mu^2 = \alpha \beta^2 \]
An important special case is when $\alpha = r/2$ where $r$ is a positive integer, and $\beta = 2$. A random variable $X$ with pdf

$$f(x) = \frac{1}{\Gamma(r/2)2^{r/2}}x^{r/2-1}e^{-x/2}$$

for $x > 0$ is said to have a **chi-square distribution** with $r$ degrees of freedom. The mgf for this distribution is

$$M(t) = (1 - 2t)^{-r/2}$$

for $t < 1/2$. 


**Example:** Let $X$ have the pdf

$$f(x) = 1$$

for $0 < x < 1$. Let $Y = -2\ln(X)$. Then $x = g^{-1}(y) = e^{-y/2}$.

The space $\mathcal{A}$ is $\{x : 0 < x < 1\}$, which the one-to-one transformation $y = -2\ln(x)$ maps onto $\mathcal{B}$.

$\mathcal{B} = \{y : 0 < y < \infty\}$.

The Jacobian of the transformation is

$$J = -\frac{1}{2}e^{-y/2}$$

Accordingly, the pdf of $Y$ is
\[ f(y) = f(e^{-y/2})|J| = \frac{1}{2}e^{-y/2} \]

for \(0 < y < \infty\).

Recall the pdf of a chi-square distribution with \(r\) degrees of freedom.

\[ f(x) = \frac{1}{\Gamma(r/2)2^{r/2}}x^{r/2-1}e^{-x/2} \]

From this we see that \(f(x) = f(y)\) when \(r = 2\).

**Definition** (Book) If \(Z\) is a standard normal random variable, the distribution of \(U = Z^2\) is called a chi-square distribution with 1 degree of freedom.

**Theorem** If the random variable \(X\) is \(N(\mu, \sigma^2)\), then the random variable \(V = (X - \mu)^2/\sigma^2\) is \(\chi^2(1)\).
Beta Distribution

Let $X_1$ and $X_2$ be independent gamma variables with joint pdf

$$h(x_1, x_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x_1^{\alpha-1} x_2^{\beta-1} e^{-x_1-x_2}$$

for $0 < x_1 < \infty$ and $0 < x_2 < \infty$, where $\alpha > 0$, $\beta > 0$.

Let $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1+X_2}$.

$$y_1 = g_1(x_1, x_2) = x_1 + x_2$$

$$y_2 = g_2(x_1, x_2) = \frac{x_1}{x_1 + x_2}$$

$$x_1 = h_1(y_1, y_2) = y_1y_2$$

$$x_2 = h_2(y_1, y_2) = y_1(1 - y_2)$$
\[ J = \begin{vmatrix} y_2 & y_1 \\ (1 - y_2) & -y_1 \end{vmatrix} = -y_1 \]

The transformation is one-to-one and maps \( \mathcal{A} \), the first quadrant of the \( x_1 x_2 \) plane onto \( \mathcal{B} = \{(y_1, y_2) : 0 < y_1 < \infty, 0 < y_2 < 1 \} \).

The joint pdf of \( Y_1, Y_2 \) is

\[
    f(y_1, y_2) = \frac{y_1}{\Gamma(\alpha)\Gamma(\beta)} (y_1y_2)^{\alpha-1} [y_1(1 - y_2)]^{\beta-1} e^{-y_1} \\
    = \frac{y_2^{\alpha-1}(1 - y_2)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} y_1^{\alpha+\beta-1} e^{-y_1}
\]

for \((y_1, y_2) \in \mathcal{B}\).

Because \( \mathcal{B} \) is a rectangular region and because \( g(y_1, y_2) \) can be factored into a function of \( y_1 \) and a function of \( y_2 \), it follows that \( Y_1 \) and \( Y_2 \) are statistically independent.
The marginal pdf of $Y_2$ is

$$f_{Y_2}(y_2) = \frac{y_2^{\alpha-1}(1 - y_2)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_{0}^{\infty} y_1^{\alpha+\beta-1} e^{-y_1} dy_1$$

$$= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} y_2^{\alpha-1}(1 - y_2)^{\beta-1}$$

for $0 < y_2 < 1$.

This is the pdf of a **beta distribution** with parameters $\alpha$ and $\beta$.

Also, since $f(y_1, y_2) = f_{Y_1}(y_1)f_{Y_2}(y_2)$ we see that

$$f_{Y_1}(y_1) = \frac{1}{\Gamma(\alpha + \beta)} y_1^{\alpha+\beta-1} e^{-y_1}$$

for $0 < y_1 < \infty$.

Thus, we see that $Y_1$ has a gamma distribution with parameter values $\alpha + \beta$ and 1.
To find the mean and variance of the beta distribution, it is helpful to notice that from the pdf, it is clear that for all $\alpha > 0$ and $\beta > 0$,

$$
\int_0^1 y^{\alpha-1}(1 - y)^{\beta-1} dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}
$$

The expected value of a random variable with a beta distribution is

$$
\int_0^1 yg(y)dy = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 y^\alpha(1 - y)^{\beta-1} dy
$$

$$
= \frac{\Gamma(\alpha + 1)\Gamma(\beta)}{\Gamma(\alpha + 1 + \beta)} \times \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)}
$$

$$
= \frac{\alpha}{\alpha + \beta}
$$

This follows from applying the fact that

$$
\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)
$$
To find the variance, we apply the same idea to find $E[Y^2]$ and use the fact that $\text{var}(Y) = E[Y^2] - \mu^2$.

$$\sigma^2 = \frac{\alpha \beta}{(\alpha + \beta + 1)(\alpha + \beta)^2}$$
t distribution

Let $W$ and $V$ be independent random variables for which $W$ is $N(0, 1)$ and $V$ is $\chi^2(r)$.

$$f(w, v) = \frac{1}{\sqrt{2\pi}} e^{-w^2/2} \frac{1}{\Gamma(r/2)2^{r/2}v^{r/2-1}} e^{-r/2}$$

for $-\infty < w < \infty$, $0 < v < \infty$.

Define a new random variable $T$ by

$$T = \frac{W}{\sqrt{V/r}}$$

To find the pdf $f_T(t)$ we use the change of variables technique with transformations

$$t = \frac{w}{\sqrt{v/r}} \text{ and } u = v.$$
These define a one-to-one transformation that maps

\[ A = \{ (w, v) : -\infty < w < \infty, 0 < v < \infty \} \]

\[ B = \{ (t, u) : -\infty < t < \infty, 0 < u < \infty \}. \]

The inverse transformations are

\[ w = \frac{t\sqrt{u}}{\sqrt{r}} \] and \[ v = u. \]

Thus, it is easy to see that

\[ |J| = \sqrt{u}/\sqrt{r} \]
By applying the change of variable technique, we see that the joint pdf of $T$ and $U$ is

$$f_{TU}(t, u) = f_{WV}(t\sqrt{\frac{u}{r}}, u)|J|$$

$$= \frac{u^{r/2-1}}{\sqrt{2\pi r\Gamma(r/2)2^{r/2}}} \exp \left[-\frac{u}{2}(1 + t^2/r)\right] \frac{\sqrt{u}}{\sqrt{r}}$$

for $-\infty < t < \infty$, $0 < u < \infty$.

To find the marginal pdf of $T$ we compute

$$f_T(t) = \int_0^\infty f(t, u)du$$

$$= \int_0^\infty \frac{u^{(r+1)/2-1}}{\sqrt{2\pi r\Gamma(r/2)2^{r/2}}} \exp \left[-\frac{u}{2}(1 + t^2/r)\right] du$$

This simplifies with a change of variables $z = u[1 + (t^2/r)]/2$. 
\[ f_T(t) = \int_0^\infty \frac{1}{\sqrt{2\pi r}\Gamma(r/2)2^{r/2}} \left( \frac{2z}{1 + t^2/r} \right)^{(r+1)/2-1} e^{-z} \left( \frac{2}{1 + t^2/r} \right) dz \]

\[ = \frac{\Gamma[(r + 1)/2]}{\sqrt{\pi r}\Gamma(r/2)(1 + t^2/2)^{(r+1)/2}} \]

for \(-\infty < t < \infty\).

A random variable with this pdf is said to have a \textbf{t distribution} with \(r\) \textbf{degrees of freedom}. 
F Distribution

Let $U$ and $V$ be independent chi-square random variables with $r_1$ and $r_2$ degrees of freedom, respectively.

$$f(u, v) = \frac{u^{r_1/2-1}v^{r_2/2-1}e^{-(u+v)/2}}{\Gamma(r_1/2)\Gamma(r_2/2)2^{(r_1+r_2)/2}}$$

Define a new random variable

$$W = \frac{U/r_1}{V/r_2}$$

To find $f_W(w)$ we consider the transformation

$$w = \frac{u/r_1}{v/r_2} \text{ and } z = v.$$  

This maps

$A = \{(u, v) : 0 < u < \infty, 0 < v < \infty\}$ to $B = \{(w, z) : 0 < w < \infty, 0 < z < \infty\}$.
The inverse transformations are

\[ u = (r_1/r_2)zw \text{ and } v = z. \]

This results in

\[ |J| = (r_1/r_2)z \]

The joint pdf of \( W \) and \( Z \) by the change of variables technique is

\[
f(w, z) = \frac{(r_1zw)^{r_1/2-1}z^{r_2/2-1}}{\Gamma(r_1/2)\Gamma(r_2/2)2^{(r_1+r_2)/2}} \exp \left[ -\frac{z}{2} \left( \frac{r_1w}{r_2} + 1 \right) \right] \frac{r_1z}{r_2}
\]

for \((w, z) \in \mathcal{B}\).

The marginal pdf of \( W \) is

\[
f_W(w) = \int_0^\infty f(w, z) dz
\]
We simplify this by changing the variable of integration to
\[ y = \frac{z}{2} \left( \frac{r_1 w}{r_2} + 1 \right) \]

Then the pdf \( f_W(w) \) is
\[
\int_0^\infty \frac{(r_1/r_2)^{r_1/2}(w)^{r_1/2-1}z^{r_1+r_2/2-1}}{\Gamma(r_1/2)\Gamma(r_2/2)2^{(r_1+r_2)/2}} \exp \left[ -\frac{z}{2} \left( \frac{r_1 w}{r_2} + 1 \right) \right] \, dz
\]

\[
= \frac{\Gamma((r_1 + r_2)/2)(r_1/r_2)^{r_1/2}(w)^{r_1/2-1}}{\Gamma(r_1/2)\Gamma(r_2/2)(1 + r_1 w/r_2)^{(r_1+r_2)/2}}
\]

for \( 0 < w < \infty \).

A random variable with a pdf of this form is said to have an **F-distribution** with **numerator degrees of freedom** \( r_1 \) and **denominator degrees of freedom** \( r_2 \).