A Derivation: The Bivariate Normal Distribution

Let Z_1 and Z_2 be independent $N(0, 1)$ random variables.

Define new random variables:

$$X = a_X Z_1 + b_X Z_2 + c_X$$
$$Y = a_Y Z_1 + b_Y Z_2 + c_Y$$

Define new constants:

$$a_X = \sqrt{(1+\rho)/2}\sigma_X, b_X = \sqrt{(1-\rho)/2}\sigma_X, c_X = \mu_Y$$
$$a_Y = \sqrt{(1+\rho)/2}\sigma_Y, b_Y = -\sqrt{(1-\rho)/2}\sigma_Y, c_Y = \mu_Y$$

and

$$E(X) = \mu_X, Var(X) = \sigma_X^2$$
$$E(Y) = \mu_Y, Var(Y) = \sigma_Y^2$$
$$\rho_{XY} = \rho$$

Let $D = a_X b_Y - a_Y b_X = -\sqrt{1-\rho^2}\sigma_X\sigma_Y$

Solve for Z_1 and Z_2.

$$Z_1 = \frac{\sigma_Y(X-\mu_X)+\sigma_X(Y-\mu_Y)}{\sqrt{2(1+\rho)}\sigma_X\sigma_Y}$$

$$Z_2 = \frac{\sigma_Y(X-\mu_X)+\sigma_X(Y-\mu_Y)}{\sqrt{2(1-\rho)}\sigma_X\sigma_Y}$$

Also, $J = 1/D = \frac{1}{-\sqrt{1-\rho^2}\sigma_X\sigma_Y}$

and we have

$$f_{XY}(x,y) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}Z_1^2)\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}Z_2^2)(\frac{1}{\sqrt{1-\rho^2}\sigma_X\sigma_Y})$$

$$= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}-\frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right)$$

$-\infty < x < \infty, -\infty < y < \infty$

a bivariate normal pdf!

Section 3.3, Example F Bivariate Normal Density

$f_{XY}(x, y)$ is constant if

$$\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X \sigma_Y} = \text{constant}$$

The locus of such points is an ellipse centered at (μ_X, μ_y).

Section 3.3 Example F Marginal Density of X, $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$

Make a change of variables, $u = (x - \mu_X)/\sigma_X$ and $v = (y - \mu_Y)/\sigma_Y$,then

$$f_X(x) = \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2(1-\rho^2)}(u^2 + v^2 - 2\rho uv)\right] dv$$

Let's complete the square to evaluate the integral,

$$u^2 + v^2 - 2\rho uv = (v - \rho u)^2 + u^2(1 - \rho^2)$$

Now,

$$f_X(x) = \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \exp(-u^2/2) \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2(1-\rho^2)}(v - \rho u)^2\right] dv$$

Recognize the integral?

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp(-(1/2)[(x - \mu_x)^2/\sigma_X^2])$$

Section 3.5 Example C Conditional Density of Y given $X = x$.

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

After some messy algebra, we see this is a normal density with

mean $\mu_Y + \rho(x - \mu_X)\sigma_Y/\sigma_X$ and variance $\sigma_Y^2(1 - \rho^2)$

Conditional mean is

$$E(Y|X) = \mu_Y + \rho(X - \mu_X)\sigma_Y/\sigma_X$$

What about mgf of a bivariate normal distribution? $M(t_1, t_2) = ...$ try it!

4.3 Covariance and Correlation

p.130-131 Develop Expressions for linear combinations of random variables.

$Cov(a + X, Y) = Cov(X, Y)$
$Cov(aX, bY) = abCov(X, Y)$
$Cov(X, Y + Z) = Cov(X, Y) + Cov(X, Z)$
$Cov(aW+bX, cY+dZ) = acCov(W, Y)+bcCov(X, Y)+adCov(W, Z)+bdCov(X, Y)$

Theorem A Suppose that $U = a + \Sigma_{i=1}^{n} b_i X_i$ and $V = c + \Sigma_{j=1}^{m} d_j Y_j$, Then

$$Cov(U, V) = \sum_{i=1}^{n} \sum_{j=1}^{m} b_i d_j Cov(X_i, Y_j)$$

Corollary A

$$Var(a + \sum_{i=1}^{n} b_i X_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_i b_j Cov(X_i, X_j)$$

Example BVN $Cov(X, Y) = a_x a_y + b_x b_y$

If the X_i are independent, then $Cov(X_i, Y_i) = 0$ for $i \neq j$, and

Corollary B $Var(\Sigma_{i=1}^n X_i) = \Sigma_{i=1}^n Var(X_i)$, if the X_i are independent.

What about expectations?

4.4 Conditional Expectation and Prediction

Recall, the conditional expectation of Y given $X = x$ is

$$E(Y|X = x) = \Sigma_y \, y p_{Y|X}(y|x) \text{ (discrete case)}$$

and

$$E(Y|X = x) = \int y f_{Y|X}(y|x) dy \text{ (cont. case)}$$

and for a function $h(Y)$,

$$E(h(Y)|X = x) = \int h(y) f_{Y|X}(y|x) dy$$

Example D Random Sums: $T = \Sigma_{i=1}^{N} X_i$ where N is a RV with finite expectation and the X_i are RVs that are independent of N.

Using Theorem A: $E(Y) = E[E(Y|X)]$ then

$E(T) = E[E(Y|N))$.

Since $E(T|N = n) = nE(X), E(T|N) = NE(X)$ and then

$$E(T) = E[NE(X)] = E(N)E(X)$$

Example E Random Sums: $T = \Sigma_{i=1}^{N} X_i$ with additional assumption that the X_i are independent RVs with the same mean, $E(X)$, and the same variance $V(X)$, and that $Var(N) < \infty$.

Using Theorem B: $Var(Y) = Var[E(Y|X)] + E[Var(Y|X)]$, then

$$Var(T) = [E(X)]^2 Var(N) + E(N)Var(X)$$

Properties of Moment Generating Functions

Property C If X has the mgf $M_X(t)$ and $Y = a + bX$, then Y has the mgf $M_Y(t) = \exp(at)M_X(bt)$.

Proof?

Property D If X and Y are independent mgf's M_X and M_Y and $Z = X + Y$, then $M_Z(t) = M_X(t)M_Y(t)$ on the common interval where both mgf's exist.

4.4.2 Prediction and Mean Squared Error

1. Predict Y by a constant value c.

$MSE = E[(Y - c)^2]$

Find the value of c that minimizes MSE.

2. Predict Y by some function $h(X)$

minimize $MSE = E[(Y - h(X))^2]$

Example $h(x) = \alpha + \beta x$ linear function

minimize $MSE = E[(Y - \alpha - \beta X)^2]$