Chapter 4 Expected Values

4.1 The Expected Value of a Random Variables

Definition. Let X be a random variable having a pdf $f(x)$. Also, suppose the the following conditions are satisfied:

$\sum |x| f(x)$ converges to a finite limit (discrete case)

$\int_{-\infty}^{\infty} |x| f(x) dx$ converges to a finite limit (continuous case)

Then the **expected value** of X is given by

$$E[X] = \sum x f(x)$$

in the discrete case, and

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

in the continuous case.

Note that the expected value does not always exists.

For instance, consider a discrete variable with space $\{1, 2, 3., , , , \}$ and $f(n) = c/n^2$ for a constant c that allows f to sum to 1. Then

$$\sum_{n=1}^{\infty} nf(n) = \sum_{n=1}^{\infty} c/n$$

which does not converge.

Book Examples Expectation of:
- Geometric Random Variable
- Poisson Distribution
- Gamma Density
- Normal Distribution
- Cauchy Density

Examples

4.1.1 Expectations of Functions of Random Variables

Consider a continuous increasing function of X, $Y = g(X)$.

$$F_Y(y) = P[Y \leq y] = P[g(x) \leq y]$$

$$= P[X \leq g^{-1}(y)] = F(g^{-1}(y)) = \int_{\infty}^{g^{-1}(y)} f(x)dx.$$

where $f(x)$ is the pdf of X.

Let f_Y denote the pdf of Y. To find this we obtain

$$F_Y'(y) = f_Y(y) = F'[g^{-1}(y)]g^{-1\prime}(y)$$

Then we have

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy$$

How does $E[Y]$ compare to the integral I

$$I = \int_{-\infty}^{\infty} g(x)f(x)dx$$

By applying a change of variables with $x = g^{-1}(y)$, have have

$$\frac{dx}{dy} = g^{-1\prime}(y) > 0$$

and

$$I = \int_{-\infty}^{\infty} yf[g^{-1}(y)]g^{-1\prime}(y)dy$$

$$= \int_{-\infty}^{\infty} yf_Y(y)dy$$

Examples.

Example 1 The following result holds true for continuous random variables that take only nonnegative values.

$$E[X] = \int_0^\infty P[X > t]dt = \int_0^\infty (1 - F(t))dt$$

Let's verify this with the uniform distribution.

X has pdf $f(x) = 1$ for $x \in (0, 1)$ and $f(x) = 0$ elsewhere. Then

$$E[X] = \int_0^1 xdx = 1^2/2 - 0^2/2 = 1/2$$

and

$$F(x) = \int_0^x dt = x$$

for $x \in (0, 1)$. Now

$$\int_0^1 [1 - F(x)]dx = \int_0^1 (1 - x)dx = 1 - 1^2/2 = 1/2.$$

Example 2 Let X have a discrete uniform distribution on the integers $\{51, 52, 53, ..., 100\}$. Approximate $E[1/X]$ Note that $f(x) = 1/50$ on this space.

$$E[1/X] = \sum_{x=51}^{100} \frac{1}{50x}$$

However,

$$\frac{1}{50}\int_{50}^{100} \frac{1}{x+1}dx \leq \sum_{x=51}^{100} \frac{1}{50x} \leq \frac{1}{50}\int_{50}^{100} \frac{1}{x}dx$$

$$(1/50)[ln[101] - ln[51]] = 0.0136659 <$$

$$\sum_{x=51}^{100} \frac{1}{50x} = 0.0137 <$$

$$(1/50)[ln[100] - ln[50]] = 0.01386294$$

Some Special Expectations: 4.1.2, 4.2, 4.5

$E[X]$ is often called the **mean value** or **mean** of X. It serves as the primary moment-based measure of the location of a distribution, and is often denoted by μ.

The dispersion of a distribution is often described by the **second central moment** more commonly known as the **variance**.

$$E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x)$$

in the discrete case.

$$E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$$

in the continuous case. It is common to let the notation σ^2 denote the variance.

$$\sigma^2 = E[(X-\mu)^2] = E[X^2] - 2\mu E[X] + \mu^2 = E[X^2] - \mu^2$$

The **standard deviation** of a random variable X is defined to be the square root of its variance, and is often denoted by σ.

Examples in Book for:

Bernoulli Distribution

Normal Distribution

Uniform Distribution

Theorem (Markov): Let $g(X)$ be a nonnegative function of a random variable X. If $E[g(X)]$ exists, then for every positive c,

$$P[g(X) \geq c] \leq \frac{E[g(X)]}{c}$$

A proof is given below for the continuous case.

Proof: Let $A = \{x : g(x) \geq c\}$, and let $f(x)$ be the pdf of X.

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

$$= \int_A g(x)f(x)dx + \int_{A^c} g(x)f(x)dx$$

From this and the nonnegativity of $g(X)$ we have

$$E[g(X)] \geq \int_A g(x)f(x)dx \geq \int_A cf(x)dx$$

which implies

$$\frac{E[g(X)]}{c} \geq P(A) = P[g(X) \geq c]$$

A special case of Markov's inequality is **Chebyshev's Inequality**.

Theorem (Chebyshev): Let X be a random variable with a finite variance σ^2. Then for every $k > 0$

$$P[|X - \mu| \geq k\sigma] \leq \frac{1}{k^2}$$

Proof: In the previous theorem take $g(X) = (X - \mu)^2$ and let $c = k^2\sigma^2$ and the proof follows immediately.

Example 3: Suppose that it is known that the number of items produced in a factory during a week is a random variable with mean 50.

a) What can be said about the probability that a week's production will exceed 75? By Markov's inequality

$$P[X \geq 75] \leq E[X]/75 = 50/75 = 2/3.$$

b) If the variance of the week's production is known to equal 25, then what can be said about the probability that this week's production will be between 40 and 60.

$$P[|X - 50| \geq 10] = P[|X - 50| \geq 2\sqrt{25}] \leq 1/4$$

Hence

$$P[|X - 50| < 10] \geq 1 - 1/4 = 3/4$$

Because Chebyshev's inequality is valid for all distributions with a variance, it cannot always be expected to give the most precise bound. Consider the example below.

Example 4 Suppose X had pdf $f(x) = \frac{1}{10}$ on the interval $(0, 10)$. Then $\mu = 5$ and $\sigma^2 = 25/3$.

$$P[|X - 5| \geq 4] = 1 - \int_1^9 \frac{1}{10} dx = 1 - 0.8 = 0.2$$

However, if we use Chebyshev's inequality

$$P[|X - 5| \geq 4] \leq \frac{25}{3(16)} \approx 0.52.$$

4.5 The Moment-Generating Function

Another special expectation is $E[e^{tX}]$ which is a function of t, and is known as the **moment-generating function** $M(t)$.

Suppose that there is a positive number h such that for $t \in (-h, h)$ $E[e^{tX}]$ exists. Then, in the continuous case

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

and in the discrete case

$$M(t) = \sum_{x} e^{tx} f(x).$$

Not every random variable has a moment-generating function. However, for those that do, the moment generating function completely determines the distribution. If two random variables have the same mgf, they must have identical distributions. The existence of $M(t)$ for $t \in (-h, h)$ implies the existence of derivatives of $M(t)$ of all orders at $t = 0$.

Furthermore, by a theorem that allows us to change the order of differentiation and integration, we have

$$\frac{dM(t)}{dt} = M'(t) = \int_{-\infty}^{\infty} xe^{tx} f(x)dx$$

or for discrete variables

$$M'(t) = \sum_x xe^{tx} f(x)$$

Thus, it is clear from the definition of $E[X]$ that $M'(0) = E[X] = \mu$.

In general, we can see that $M''(0) = E[X^2]$, $M'''(0) = E[X^3]$, and so on.

Book Examples mgf of:
- Poisson Distribution
- Gamma Distribution
- Normal Distribution

Examples.