3.6 Functions of Jointly Distributed Random Variables

Discrete Random Variables:

Let $f(x, y)$ denote the joint pdf of random variables X and Y with $\mathcal{A}$ denoting the two-dimensional space of points for which $f(x, y) > 0$.

Let $u = g_1(x, y)$ and $v = g_2(x, y)$ define a one-to-one transformation that maps $\mathcal{A}$ onto the space of U and V, $\mathcal{B}$.

The joint pdf of $U = g_1(X, Y)$ and $V = g_2(X, Y)$ is $f_{UV}(u, v)$ for $(u, v) \in \mathcal{B}$, where $x = h_1(u, v)$, $y = h_2(u, v)$ is the inverse of $u = g_1(x, y)$, $v = g_2(x, y)$

Example 1: Let X and Y be two independent random variables that have Poisson distributions with means μ_1 and μ_2, respectively.

$$f(x, y) = \frac{\mu_1^x \mu_2^y e^{-\mu_1 - \mu_2}}{x!y!}$$

for $x = 0, 1, 2, ...,$ and $y = 0, 1, 2, ...,.$
The space $\mathcal{A}$ of points (x, y) such that $f(x, y) > 0$, is just all pairs of nonnegative integers.

We want to find the pdf of $U = X + Y$. It will help to use the change of variables technique. This requires defining a second transformation V, so that a one-to-one transformation between pairs (x, y) and (u, v) is created.

Define $V = Y$. Then $u = x + y$ and $v = y$, represent a one-to-one transformation that maps $\mathcal{A}$ onto

$\mathcal{B}$=$\{(u, v) : v = 0, 1, ..., u$ and $u\} = 0, 1, 2, ...\}$.

For $(u, v) \in \mathcal{B}$, the inverse functions are given by $x = u - v$ and $y = v$. Then,

$$f(u, v) = \frac{\mu_1^{u-v} \mu_2^{v} e^{-\mu_1 - \mu_2}}{(u - v)! v!}$$

From this, we can find the marginal pdf of U

$$f_U(u) = \sum_{v=0}^{u} f(u, v)$$

$$= \frac{e^{-\mu_1 - \mu_2}}{u!} \sum_{v=0}^{u} \frac{u!}{(u - v)! v!} \mu_1^{u-v} \mu_2^{v}$$

$$= \frac{(\mu_1 + \mu_2)^u e^{-\mu_1 - \mu_2}}{u!}$$

$u = 0, 1, 2, \ldots.$

From this we can see that $U = X + Y$ is Poisson with mean $\mu_1 + \mu_2$.

Continuous Random Variables:

Let $u = g_1(x, y)$ and $v = g_2(x, y)$ define a one-to-one transformation that maps a two-dimensional set $\mathcal{A}$ in the xy plane onto a (two-dimensional) set $\mathcal{B}$ in the uv plane.

If we express x and y in terms of u and v we have

$x = h_1(u, v)$ and $y = h_2(u, v)$.

The determinant of order 2,

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

is called the **Jacobian** of the transformation is a function of (u, v). We'll assume that these first-order derivatives are continuous, and the Jacobian J is not identical to 0 in $\mathcal{A}$.

Example 2: Let $\mathcal{A}$ be the square $\mathcal{A}$=$\{(x, y) : 0 < x < 1, 0 < y < 1\}$.

Consider the transformation

$$u = g_1(x, y) = x + y$$

$$v = g_2(x, y) = x - y$$

To apply the Jacobian of the transformation we first find the inverse transformation.

$$x = h_1(u, v) = \frac{1}{2}(u + v)$$

$$y = h_2(u, v) = \frac{1}{2}(u - v)$$

To determine $\mathcal{B}$ in the uv plane, note how the boundaries of $\mathcal{A}$ are transformed into the boundaries of $\mathcal{B}$.

$x = 0$ into $0 = \frac{1}{2}(u + v)$
$x = 1$ into $1 = \frac{1}{2}(u + v)$
$y = 0$ into $0 = \frac{1}{2}(u - v)$
$y = 1$ into $1 = \frac{1}{2}(u - v)$.

The Jacobian is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial x_2}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

Note: Your book calls this $J^{-1}(h_1(u, v), h_2(u, v))$ and uses the notation $J(x, y)$.

Still need to take absolute value, $|J^{-1}(h_1(u, v), h_2(u, v)|$ as in Book Proposition A.

Examples

Convolution: Finding the pdf of the sum of two independent random variables.

Let X and Y be independent random variables with respective pdfs $f_X(x)$ and $f_Y(y)$.
Let $Z = X + Y$ and $W = Y$.

We have the one-to-one transformation $x = z - w$ and $y = w$ with Jacobian $J = 1$.

Therefore the joint pdf of Z and W is

$$f_{ZW}(z, w) = f_X(z - w) f_Y(w)$$

and the marginal of $Z = X + Y$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z - w) f_Y(w) dw$$

Very useful!

3.7 Extrema and Order Statistics

Let $X_1, X_2, ..., X_n$ denote a random sample of the continuous type having a pdf $f(x)$ and cdf $F(x)$.

Let $X_{(1)}$ be the smallest of these, $X_{(2)}$ be the second smallest, and so on with $X_{(n)}$ denoting the largest.

$$X_{(1)} < X_{(2)} < X_{(3)} < ... < X_{(n)}$$

$X_{(k)}$ is called the kth **order statistic** of the sample for $i = 1, 2, ..., n$.

The joint pdf of $X_{(1)}, ..., X_{(n)}$ is given by

$$f(x_{(1)}, ..., x_{(n)}) = n! f(x_{(1)}) f(x_{(2)}) \cdots f(x_{(n)})$$

when $a < x_{(1)} < x_{(2)} < ... < x_{(n)} < b$.

This is quite consistent with intuition.

Consider, and vector $(x_1, x_2, ..., x_n)$. Because of independence we have the joint pdf of $X_1, .., X_n$ is

$$f(x_1, x_2, .., x_n) = f(x_1)f(x_2)\cdots f(x_n)$$

However, the order statistics are unaltered for all $n!$ permutations of $(x_1, x_2, ..., x_n)$, which results in the coefficient in the pdf of $X_{(1)}, X_{(2)}, ..., X_{(n)}$ above.

Next, consider the maximum of the sample $X_{(n)}$. Let $F_n(x_{(n)})$ denote the cdf of $X_{(n)}$. We find the cdf and pdf of $X_{(n)}$ written in terms of F and f.

$$P[X_{(n)} \leq x_{(n)}] = P[X_1 \leq x_{(n)}, X_2 \leq x_{(n)}, ..., X_n \leq x_{(n)}]$$

$$= \prod_{i=1}^{n} P[X_i \leq x_{(n)}] = [F(x_{(n)})]^n$$

Thus, $F_n(x_{(n)}) = [F(x_{(n)})]^n$.

We find the pdf of $X_{(n)}$ by differentiating the cdf.

$$f_n(x_{(n)}) = F_n'(x_{(n)}) = n[F(x_{(n)})]^{n-1} f(x_{(n)})$$

for $a < x_{(n)} < b$

Now consider the minimum of the sample $X_{(1)}$.

$$1 - F_1(x_{(1)}) = P[X_{(1)} > x_{(1)}] =$$

$$\prod_{i=1}^{n} P[X_i > x_{(1)}] = \prod_{i=1}^{n} [1 - F(x_{(1)})] = [1 - F(x_{(1)})]^n$$

We see that

$$F_1(x_{(1)}) = 1 - [1 - F(x_{(1)})]^n$$

and the pdf of $X_{(1)}$ is found by taking a derivative

$$f_1(x_{(1)}) = F_1'(x_{(1)}) = n[1 - F(x_{(1)})]^{n-1} f(x_{(1)}).$$

for $a < x_{(1)} < b$.

In general, suppose we are interested in the pdf of the kth order statistic , for $1 \leq k \leq n$. To find $F_k(x_{(k)})$, we notice that the probability of $\{X_{(k)} \leq x_{(k)}\}$ is just the probability that at least k of the X's are less than $x_{(k)}$

The chance that $X \leq x_{(k)}$ is $F(x_{(k)})$, so we find that

$$F_k(x_{(k)}) = P[X_{(k)} \leq x_{(k)}]$$

$$= \sum_{i=k}^{n} \frac{n!}{(n-i)!(i)!} F(x_{(k)})^i [1 - F(x_{(k)})]^{n-i}$$

and $f_k(x_{(k)})$ is just found by taking the derivative of $F_k(x_{(k)})$.

However, by applying an interesting identity in analysis, it can be shown that $f_k(x_{(k)})$ simplifies to

$$f_k(x_{(k)}) = \frac{n!}{(k-1)!(n-k)!} [F(x_{(k)})]^{k-1} [1 - F(x_{(k)})]^{n-k} f(x_{(k)})$$

The above is Theorem A in Book.