3.2 Discrete Random Variables

Definition: Given a random experiment with a sample space Ω , consider two random variables X_1 and X_2, which assign to each element in Ω one and only one ordered pair of number $X_1(\omega) = x_1$ and $X_2(\omega) = x_2$.

The space of X_1 and X_2 is the set of ordered pairs
$\mathcal{A}$=$\{(x_1, x_2) : x_1 = X_1(\omega), x_2 = X_2(\omega), \omega \in \Omega\}$.

Example 1: Consider the random experiment of flipping a coin three times. Ω=$\{\omega_1, \omega_2, ..., \omega_8\}$.
$\omega_1 = TTT$, $\omega_2 = TTH$, $\omega_3 = THT$, $\omega_4 = HTT$, $\omega_5 = THH$, $\omega_6 = HTH$, $\omega_7 = HHT$, $\omega_8 = HHH$.

Let X_1 equal the number of heads, and let X_2 equal the number of tails.
$X_1(\omega_1) = 0$, $X_1(\omega_2) = X_1(\omega_3) = X_1(\omega_4) = 1, X_1(\omega_5) = X_1(\omega_6) = X_1(\omega_7) = 2$, $X_1(\omega_8) = 3$.

Clearly, $X_2 = 3 - X_1$ for all ω.
Consider the event $A = \{(2,1)\} \subset \mathcal{A}$. Let $C = \{\omega_5, \omega_6, \omega_7\} \subset \Omega$.

$$P[(X_1, X_2) \in A] = P[C]$$

Of course, if the flips are independent and it is a fair coin

$$P[\omega_5] = P[\omega_6] = P[\omega_7] = (1/2)^3 = 1/8$$

and

$$P[C] = 1/8 + 1/8 + 1/8 = 3/8.$$

Example 2: Let $\mathcal{A}$ consists of all pairs of positive integers (x, y) for $x = 1, 2, ...$ and $y = 1, 2, ...$ Show that $f(x, y) = \frac{9}{4^{x+y}}$ is a proper probability density function for the pair (X, Y) with space $\mathcal{A}$.

Certainly $f(x, y)$ is positive on the discrete space $\mathcal{A}$. We need to show that

$$\sum_{x=1}^{\infty} \sum_{y=1}^{\infty} f(x, y) = 1$$

$$\sum_{x=1}^{\infty} \sum_{y=1}^{\infty} f(x, y) = \sum_{x=1}^{\infty} \frac{9}{4^x} \sum_{y=1}^{\infty} (1/4)^y$$

Using our knowledge that for $r \in (0, 1)$ $\Sigma_{y=1}^{\infty} r^y = r/(1 - r)$ we have,

$$\sum_{x=1}^{\infty} \frac{9}{4^x} \sum_{y=1}^{\infty} (1/4)^y = \sum_{x=1}^{\infty} \frac{9}{4^x} (1/3)$$

Now consider a pair of discrete random variables X_1 and X_2, with joint frequency function or joint pdf $p(x_1, x_2)$. We can find the **marginal** pdf of X_1, $f_{X_1}(x_1)$ or by your book notation $p_{X_1}(x_1)$,

$$p_{X_1}(x_1) = \sum_{x_2} p(x_1, x_2)$$

and similarly,

$$p_{X_2}(x_2) = \sum_{x_1} p(x_1, x_2).$$

Example Table

3. Continuous Random Variables

Example 3: Consider the pdf of a pair of random variables X, Y defined by $f(x, y) = 6x^2y$ on the square $0 < x < 1$, $0 < y < 1$. $f(x, y) = 0$ outside of this square.
Find $P[0 < X < 3/4, 1/3 < Y < 2]$

$$P(0 < X < 3/4, 1/3 < Y < 2) = \int_{1/3}^{2}\int_{0}^{3/4} f(x, y)dxdy$$

$$= \int_{1/3}^{1}\int_{0}^{3/4} 6x^2ydxdy + \int_{1}^{2}\int_{0}^{3/4} 0dxdy$$

$$= \int_{1/3}^{1}\int_{0}^{3/4} 6x^2ydxdy = \int_{1/3}^{1} y\left[2x^3|_0^{3/4}\right]dy$$

$$= 54/64\int_{1/3}^{1} ydy = 54/64\left[y^2/2|_{1/3}^{1}\right]$$

$$= 54/64(1/2 - 1/18) = 3/8$$

Note: Note this is probability is the volume under the surface $f(x, y)$ and above the rectangular set in the xy-plane.

The definition of the distribution function for two variables is the expected generalization of the definition in one dimension.

$$F(x, y) = P[X \leq x, Y \leq y]$$

If X and Y are of the continuous type and have pdf $f(x, y)$ then

$$F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(u, v) du dv$$

Also, at points of continuity of $f(x, y)$ we have

$$\frac{\partial^2 F(x, y)}{\partial x \partial y} = f(x, y)$$

When we have two random variables X_1 and X_2, it is common to call $f(x_1, x_2)$ the **joint pdf** and $F(x_1, x_2)$ the **joint distribution** or the **joint cdf**.
For a pair of continuous random variables, marginal densities are

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2) dx_2$$

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2) dx_1$$

3.5 Conditional Distributions

Recall that the conditional probability of an event A given an event B is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Let x_1 be a value such that $f_{X_1}(x_1) > 0$. Then, the **conditional pdf** of X_2 given $X_1 = x_1$ is defined by

$$f_{X_2|X_1}(x_2|x_1) = \frac{f(x_1, x_2)}{f_{X_1}(x_1)}$$

Similarly, when $f_{X_2}(x_2) > 0$ the conditional pdf of X_1 given $X_2 = x_2$ is

$$f_{X_1|X_2}(x_1|x_2) = \frac{f(x_1, x_2)}{f_{X_2}(x_2)}$$

Examples

3.4 Independent Random Variables

Definition: Let random variables X_1 and X_2 have joint cdf $F(x_1, x_2)$. The random variables X_1 and X_2 are said to be independent if and only if $F(x_1, x_2) = F_{X_1}(x_1)F_{X_2}(x_2)$.

or equivalently,

Definition: Let random variables X_1 and X_2 have joint pdf $f(x_1, x_2)$. The random variables X_1 and X_2 are said to be independent if and only if $f(x_1, x_2) = f_{X_1}(x_1)f_{X_2}(x_2)$.

Example: Suppose that X_1 and X_2 are independent and find $f_{X_2|X_1}(x_2|x_1)$.

$$f_{X_2|X_1}(x_2|x_1) = \frac{f(x_1, x_2)}{f_{X_1}(x_1)} = \frac{f_{X_1}(x_1)f_{X_2}(x_2)}{f_{X_1}(x_1)} = f_{X_2}(x_2)$$

Example