2.3 **Functions of a Random Variable**

Let X denote a random variable with density function $f(x)$, and and define $Y = g(X)$ for some function g. We want to find the pdf of $g(X)$.

There are three approaches:

1. Distribution-function technique
2. Change-of-variables or transformations technique
3. Using generating functions

We will cover 1 and 2, for now.

Distribution Function Technique

First find the cdf of Y then differentiate w.r.t Y, to obtain:

$$f(y) = \frac{d}{dy} F(y)$$

Note: Let X denote a random variable with cdf $F(x)$, and define $Y = g(X)$ for some function g. Let F_Y denote the cdf of Y.

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(A)$$

where $A = \{x : g(x) \le y\}$.

Example:

Transformations of Discrete Variables

Example: Let X have the Poisson pdf

$$f(x) = \frac{\mu^x e^{-\mu}}{x!}$$

for $x = 0, 1, 2, ...$

Then X has space $\mathcal{A}$=$\{0, 1, 2, 3, 4, ..., \}$.

Now define a new random variable $Y = 4X$. Then the space $\mathcal{A}$ maps **onto** the space of Y, which we'll denote by $\mathcal{B} = \{0, 4, 8, 12, ...\}$.

Under this map, there is a one-to-one correspondence between points in $\mathcal{A}$ and points in $\mathcal{B}$.

Define $f(y) = P(Y = y)$,

$$f(y) = P(Y = y) = P(X = y/4) = \frac{\mu^{y/4} e^{-\mu}}{(y/4)!}, \; y = 0, 4, 8, ...$$

Transformations of Continuous Variables

Example: Let X have pdf

$$f(x) = 2x$$

for $0 < x < 1$. $\mathcal{A}$=$\{x : 0 < x < 1\}$, is the space where $f(x) > 0$.

Define the random variable $Y = 8X^3$. Under the transformation $y = 8x^3$ $\mathcal{A}$ is mapped onto the space of Y, $\mathcal{B}$=$\{y : 0 < y < 8\}$. Also the mapping is one-to-one because we can find the inverse $x = \frac{y^{1/3}}{2}$.

Also, note that $\frac{dx}{dy} = \frac{1}{6y^{2/3}}$.

Let $0 < a < b < 8$.

$$P(a < Y < b) = P\left(\frac{a^{1/3}}{2} < X < \frac{b^{1/3}}{2}\right)$$

Let $l = a^{1/3}/2$ and $u = b^{1/3}/2$

$$= \int_l^u 2x dx$$

Let's rewrite this integral by changing the variable of integration to $y = 8x^3$. Then

$$P(a < Y < b) = \int_a^b 2\left(\frac{y^{1/3}}{2}\right)\left(\frac{1}{6y^{2/3}}\right) dy$$

$$= \int_a^b \frac{1}{6y^{1/3}} dy$$

Since this is true for all $0 < a < b < 1$, the pdf of Y must be the integrand,

$$f(y) = \frac{1}{6y^{1/3}}$$

for $0 < y < 8$.

Proposition B Let X be a continuous random variable with density $f(x)$ and let $Y = g(X)$ where g is differentiable, strictly monotonic function on some interval I. Suppose $f(x) = 0$ if x is not in I. Then Y has the density function

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

for y such that $y = g(x)$ for some x, and $f_Y(y) = 0$ if $y \neq g(x)$ for any x in I.

Note: It is easiest to deal with functions $g(x)$ that are *monotone*.

We actually need:
(1) The set $\mathcal{B}$ of points where $f(y) > 0$ and
(2) The integrand of the integral on y to which $P(a < Y < b)$ is equal.

So, sometimes the conditions of the Proposition B (Theorem) are:

Let X have a pdf of the continuous type and let $\mathcal{A}$ be the one-dimensional space where $f(x) > 0$. Consider the random variable $Y = g(X)$, where $y = g(x)$ defines a one-to-one transformation that maps $\mathcal{A}$ onto the set $\mathcal{B}$. Let $\frac{d}{dy}g^{-1}(y)$ be continuous and not equal to zero in $\mathcal{B}$.

Example:

Proposition C (read)

Proposition D Let the random variable U be uniform on $[0,1]$, and let $X = F^{-1}(U)$. Then the cdf of X is F.

$$P[X \leq x] = P[F^{-1}(U) \leq x] = P[U \leq F(x)]$$

Since U has a uniform distribution,

$$P[U \leq F(x)] = \int_0^{F(x)} du = F(x)$$

Thus X has distribution function F.

This is a useful theorem because it tells us how to simulate observations from chosen distributions. Many quality random number generation algorithms are available to draw from a uniform distribution on (0,1). These can give us U. Now, by applying the result given above we can obtain $X = F^{-1}(U)$.

Example: