Chapter 9 Testing Hypothesis and Assessing Goodness of Fit

9.5 Generalized Likelihood Ratio Tests

Finally, we consider the case of testing a composite null hypothesis H_0 against a composite alternative hypothesis H_A.

Let $X_1, X_2, ..., X_n$ denote n independent random variables having the probability density functions $f_i(x_i; \theta_1, \theta_2, ..., \theta_m)$, for $i = 1, 2, ..., n$.

Let Ω denote the set of all parameter points $(\theta_1, ..., \theta_m)$. Let ω be a subset of Ω. We wish to test the (simple or composite) hypothesis

$$H_0 : (\theta_1, \theta_2, ..., \theta_m) \in \omega$$

against all possible alternative hypotheses.

Define the likelihood functions

$$L(\omega) = \prod_{i=1}^{n} f_i(x_i; \theta_1, \theta_2, ..., \theta_m)$$

for $(\theta_1, \theta_2, ..., \theta_m) \in \omega$ and

$$L(\Omega) = \prod_{i=1}^{n} f_i(x_i; \theta_1, \theta_2, ..., \theta_m)$$

for $(\theta_1, \theta_2, ..., \theta_m) \in \Omega$.

Let $L(\hat{\omega})$ and $L(\hat{\Omega})$ denote the maxima of the functions, when constrained to their respective domains.

The ratio of $L(\hat{\omega})$ to $L(\hat{\Omega})$ is called the **likelihood ratio** and is denoted by

$$\lambda(x_1, ..., x_n) = \frac{L(\hat{\omega})}{L(\hat{\Omega})}$$

The **likelihood ratio test** states that H_0 is rejected if and only if

$$\lambda(x_1, ..., x_n) \leq \lambda_0,$$

where the number λ_0 satisfies the significance level,

$$\alpha = P[\lambda(X_1, ..., X_n) \leq \lambda_0; H_0].$$

It is often difficult to determine the distribution of $\lambda(X_1, ..., X_n)$ under the null hypothesis, which is required for computing λ_0.

Under certain regularity conditions, a general large sample approximation is available. In particular, the statistic

$$-2\ln[\lambda(X_1, ..., X_n)]$$

has an approximate chi-square distribution with $m - q$ degrees of freedom for large samples when H_0 is true. Here m is the dimension of the parameter space Ω, and q is the dimension of the restricted subset of the parameter space ω.

Example 1: Let X be $N(\theta_1, \theta_2)$, with $\Omega = \{(\theta_1, \theta_2) : -\infty < \theta_1 < \infty, 0 < \theta_2 < \infty\}$. We want to test

$$H_0 : \theta_1 = 0$$

versus the composite alternative

$$H_A : \theta_1 \neq 0$$

Thus, $\omega = \{(\theta_1, \theta_2) : \theta_1 = 0, 0 < \theta_2 < \infty\}$.

We can think of Ω as a 2-dimensional space, and ω as a 1-dimensional subspace.

$$L(\Omega) = \left(\frac{1}{2\pi\theta_2}\right)^{n/2} \exp\left[-\frac{\Sigma_{i=1}^{n}(x_i - \theta_1)^2}{2\theta_2}\right]$$

and

$$L(\omega) = \left(\frac{1}{2\pi\theta_2}\right)^{n/2} \exp\left[-\frac{\Sigma_{i=1}^{n} x_i^2}{2\theta_2}\right].$$

By setting the derivative of $\ln[L(\omega)]$ with respect to θ_2 equal to 0, we find the the mle of θ_2, when $\theta_1 = 0$, is $\Sigma_{i=1}^n x_i^2/n$.

This implies that

$$L(\hat{\omega}) = \left(\frac{1}{2\pi\,\Sigma_{i=1}^n x_i^2/n}\right)^{n/2} \exp\left[-\frac{\Sigma_{i=1}^n x_i^2}{2\,\Sigma_{i=1}^n x_i^2/n}\right]$$

$$= \left(\frac{ne^{-1}}{2\pi\,\Sigma\, x_i^2}\right)^{n/2}$$

Without the restriction that $\theta_1 = 0$ we find that the mle $(\hat{\theta}_1, \hat{\theta}_2)$ is given by

$$\hat{\theta}_1 = \bar{x}$$

and

$$\hat{\theta}_2 = \sum_{i=1}^n (x_i - \bar{x})^2/n$$

Thus

$$L(\hat{\Omega}) = \left[\frac{1}{2\pi\,\Sigma(x_i - \bar{x})^2/n}\right]^{n/2} \exp\left[-\frac{\Sigma(x_i - \bar{x})^2}{2\,\Sigma(x_i - \bar{x})^2/n}\right]$$

$$= \left[\frac{ne^{-1}}{2\pi\,\Sigma(x_i - \bar{x})^2}\right]^{n/2}$$

and

$$\lambda = \left[\frac{\Sigma(x_i - \bar{x})^2}{\Sigma\, x_i^2}\right]^{n/2}$$

We reject H_0 if and only if $\lambda \le \lambda_0$, where λ_0 satisfies

$$P[\lambda(X_1, ..., X_n) \le \lambda_0; H_0] = \alpha.$$

It can be shown that $\lambda \leq \lambda_0$ if and only if

$$\frac{\sqrt{n}|\bar{x}|}{\sqrt{\Sigma(x_i - \bar{x})^2/(n-1)}} \geq \sqrt{(n-1)(\lambda_0^{-2/n} - 1)}$$

From this, we see the the likelihood ratio test is equivalent to a t-test in which we reject the null hypothesis if

$$|t(x_1, .., x_n)| \geq t_{\alpha/2}$$

where

$$t(X_1, ..., X_n) = \frac{\sqrt{n}\bar{X}}{\sqrt{\Sigma(X_i - \bar{X})^2/(n-1)}}$$

has a t-distribution with $n-1$ degrees of freedom under the null hypothesis.