5.2 The Law of Large Numbers

Definition A sequence of random variables $X_1, X_2, X_3, \ldots$ **converges in probability** to a random variable X if, for every $\epsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| \leq \epsilon) = 1,$$

or equivalently,

$$\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0.$$

We are most often interested in cases where X equal some constant c

Note: In general, **convergence in probability** is stronger than **convergence in distribution**. This means that convergence in probability implies convergence in distribution, but the reverse does not always hold.

However, in the case of convergence to constant random variables, they are equivalent.

Theorem A Law of Large Numbers (Weak Law of Large Numbers):

Let $X_1, X_2, \ldots, X_i, \ldots$ be a sequence of independent random variables with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$. Let $\overline{X}_n$ denote the sample mean of a random sample of size n Then $\overline{X}_n$ converges in probability to μ.

Proof: The mean and variance of $\bar{X}_n$ are μ and σ^2/n. Let $\epsilon > 0$. From here we just apply Chebyshev's inequality.

$$P[|\bar{X}_n - \mu| > \epsilon] \leq \frac{Var(\bar{X}_n)}{\varepsilon^2}$$

$$= \frac{\sigma^2}{n\varepsilon^2}$$

which converges to 0 as n approaches infinity.

Thus,

$$\lim_{n\to\infty} P[|\bar{X}_n - \mu| > \varepsilon] \leq \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} = 0$$

Let's illustrate the weak law of large numbers (WLLN) with a little simulation.

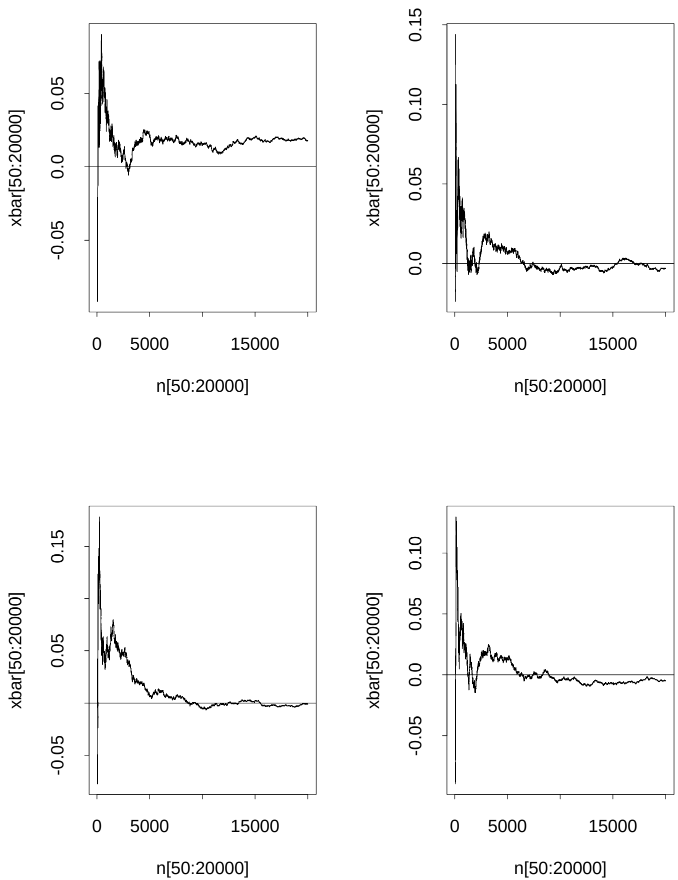
```
## Generate 20000 draws from a standard
## normal distribution and consider the
## sequence of sample means for n=1:20000
## do 4 independent experiments
par(mfrow=c(2,2))
for (i in 1:4){
z<-rnorm(20000,0,1)
n<-c(1:20000)

## Find cumulative sum and dive by sample size
xbar<-cumsum(z)/n

## Plot the vector of 20000 sample means vs. the sample size
plot(n[50:20000],xbar[50:20000],type="l")

## draw horizontal line at y=0
abline(h=0)
}
```

Figure 1: $\bar{X}_n$ versus n in 4 experiments



Homework Problem 5.4.7

Show that if $X_n \to c$ in probability and if g is a continuous function, then $g(X_n) \to g(c)$ in probability.

In order to prove this it helps to have a definition of continuity.

A function $g(x)$ is continuous at x_0 if for any $\epsilon > 0$ there is a $\delta > 0$ (which may depend on x_0) such that

$$|g(x) - g(x_0)| < \epsilon$$

whenever

$$|x - x_0| < \delta$$