6.3 The Sample Mean and the Sample Variance

Let $X_1, X_2, ..., X_n$ denote a random sample of size $n \geq 2$ from a distribution that is $N(\mu, \sigma^2)$. Here we study the distributions of the statistics

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

and

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2$$

Note: $\overline{X}$ is a linear combination of indendent normal random variables, it is normally distributed with

$$E(\overline{X}) = \mu \text{ and } Var(\overline{X}) = \frac{\sigma^2}{n}$$

We want to show

Corollary A $\overline{X}$ and S^2 are independentl distributed.

Theorem B The distribution of $(n-1)S^2/\sigma^2$ is the chi-square distribution with $n-1$ degrees of freedom.

Corollary B Let $\overline{X}$ and S^2 be as given at the beginning of this section. Then

$$\frac{\overline{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$$

Turns out we need Theorem A to prove Corollary A

Theorem A The random variable $\overline{X}$ and the random variables $(X_1 - \overline{X}, X_2 - \overline{X}, \ldots, X_n - \overline{X})$ are independent.

Let's prove some theorems!