Extension to Several Random Variables (NOT IN BOOK)

Definition: Consider a random experiment with sample space $\mathcal{C}$. Let random variable X_i assign each $c \in \mathcal{C}$ a real number $X_i(c) = x_i$, $i = 1, 2, ..., n$. The space of these n random variables is
$\mathcal{A}=\{(x_1, x_2, ..., x_n) : x_1 = X_1(c), ..., x_n = X_n(c), c \in \mathcal{C}\}$.

The distribution of $X_1, ..., X_n$ is defined by

$$F(x_1, x_2, ..., x_n) = P[X_1 \leq x_1, X_2 \leq x_2, .., X_n \leq x_n]$$

Example 1 Let $f(x, y, z) = e^{-x-y-z}$ for $0 < x, y, z < \infty$, and find $F(x, y, z)$.

$$F(x, y, z) = \int_0^z \int_0^y \int_0^x e^{-u-v-w} dudvdw$$

$$= (1 - e^{-x})(1 - e^{-y})(1 - e^{-z})$$

for $0 \leq x, y, z < \infty$.

Compute $P[X < Y < Z]$.

$$P[X < Y < Z] = \int_0^\infty \int_0^z \int_0^y e^{-x-y-z} dx dy dz$$

$$= \int_0^\infty \int_0^z e^{-y-z} - e^{-2y-z} dy dz$$

$$= \int_0^\infty \frac{e^{-3z}}{2} + \frac{e^{-z}}{2} - e^{-2z} dz = 1/6$$

In the continuous case, if we happen to know F but not the joint probability density function f, we can find f by differentiating F once with respect to each of its n components. In the case of $n = 3$

$$\frac{\partial^3 F(x,y,z)}{\partial x \partial y \partial z} = f(x,y,z)$$

Let $X_1, X_2, .., X_n$ have joint pdf f and let $g(X_1, X_2, ..., X_n)$ be a function of these variables such that the n-fold integral

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, ..., x_n) f(x_1, ..., x_n) dx_1 \cdots dx_n$$

exists, or in the discrete case

$$\sum_{x_n} \cdots \sum_{x_1} g(x_1, ..., x_n) f(x_1, ..., x_n)$$

exists. Then the n-fold integral or sum is called the **expectation**, denoted by $E[g(X_1, X_2, ..., X_n)]$.

As in the case of $n = 2$, we can find the marginal pdfs of the random variables by intgrating the joint density over the space of the remaining variables conditional on each value of the variable of interest.

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, ..., x_n) dx_2 \cdots dx_n$$

Also, when $f_{X_1}(x_1) > 0$ we can define the conditional pdf of the remaining variables given x_1.

$$f_{X_2,...,X_n|X_1}(x_2, ..., x_n|x_1) = \frac{f(x_1, x_2, ..., x_n)}{f_1(x_1)}$$

The random variables are **mutually independent** if and only if

$$f(x_1, x_2, ..., x_n) = f_1(x_1) f_2(x_2) \cdots f_n(x_n)$$

Mutual independence implies pairwise independence. Does pairwise independence imply mutual independence? Consider the following example.

Example 2 Let $f(x_1, x_2, x_3) = 1/4$ for $(x_1, x_2, x_3) \in \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$.

Then, for $i \neq j$

$$f_{i,j}(x_i, x_j) = 1/4$$

for $(x_i, x_j) \in \{(0,0), (1,0), (0,1), (1,1)\}$

and $f_{i,j}(x_i, x_j) = 0$ on all other pairs.

Also, it is easy to see that all marginal pdfs are

$f_i(x_i) = 1/2$ for $x_i = 0, 1$

Clearly, for $i \neq j$ $f_{i,j}(x_i, x_j) = f_i(x_i) f_j(x_j)$ but it is not the case that

$$f(x_1, x_2, x_3) = f_1(x_1) f_2(x_2) f_3(x_3)$$

For instance $f(1, 1, 1) = 1/4$
but $f_1(1) f_2(1) f_3(1) = 1/8$.

Definition Let the expectation

$$M(t_1, t_2, ..., t_n) = E[exp(t_1 X_1 + t_2 X_2 + \cdots + t_n X_n)]$$

exist for all $(t_1, ..., t_n)$ in a neighborhood of the origin. $M(t_1, t_2, ..., t_n)$ is the moment generating function for the joint distribution of $X_1, X_2, ..., X_n$.

Example 3: Let X, Y, Z have joint pdf $f(x, y, z) = 2(x + y + z)/3$ for $0 < x, y, z < 1$.

a. Find the conditional distribution of X and Y given $Z = z$.

First we find $f_Z(z)$.

$$f_Z(z) = 2/3 \int_0^1 \int_0^1 (x + y + z)dxdy$$

$$= \frac{2}{3} \int_0^1 1/2 + y + zdy = \frac{2}{3}(1/2 + z + 1/2) = 2(z + 1)/3$$

Just to verify this is a density we compute

$$\int_0^1 f_z(z)dz = \frac{2}{3} \int_0^1 z + 1dx = (2/3)(3/2) = 1$$

Now back to the problem. We have found the marginal density of z, so the conditional density is given by

$$f_{XY|Z}(x, y|z) = \frac{f(x, y, z)}{f_Z(z)} = \frac{x + y + z}{z + 1}$$

For $0 < x, y, z < 1$.

Just to be careful again, we'll verify that this is a density.

$$\int_0^1 \int_0^1 f_{XY|Z}(x, y|z) dx dy = \int_0^1 \int_0^1 \frac{x + y + z}{z + 1} dx dy$$

$$= \frac{1}{z + 1} \int_0^1 1/2 + y + z dy = \frac{1}{z + 1}(z + 1) = 1$$

(b) Find the conditional pdf of X given Y and Z.

$$f_{YZ}(y,z) = \frac{2}{3}\int_0^1 x + y + z dx = \frac{2}{3}(1/2 + y + z)$$

Thus, the conditional is given by

$$f_{X|Y,Z}(x|y,z) = \frac{x + y + z}{\frac{1}{2} + y + z}$$

Now find $E[X|Y,Z]$.

$$E[X|Y,Z] = \int_0^1 x f_{X|Y,Z}(x|y,z)dx$$

$$= \frac{1}{1/2 + y + z}\int_0^1 x^2 + x(y+z)dx$$

$$= \frac{1}{1/2 + y + z}[1/3 + (y+z)/2]$$

Example 4: With the random variables of Example 1, find $P[X < Y < Z|Z < 1]$

$$P[X < Y < Z|Z < 1] = \frac{P[\{X < Y < Z\} \cap \{Z < 1\}]}{P[Z < 1]}$$

By the fact the the joint pdf factors, we see that $f_z(z) = e^{-z}$ on the interval $(0, \infty)$. Thus,

$$P[Z < 1] = \int_0^1 e^{-z} dz = 1 - e^{-1}$$

By taking advantage of some work done in Example 1, we see that

$$P[\{X < Y < Z\} \cap \{Z < 1\}] = \int_0^1 \frac{e^{-3z}}{2} + \frac{e^{-z}}{2} - e^{-2z} dz$$

$$= 1/6 - 1/6e^{-3} - 1/2e^{-1} + 1/2e^{-2}$$

Thus,

$$P[X < Y < Z|Z < 1] = \frac{1/6 - 1/6e^{-3} - 1/2e^{-1} + 1/2e^{-2}}{1 - e^{-1}} \approx 0.0665$$

Let's do a little computer simulation to check that
no calculus error was made above.

```
> x<-rexp(100000)
> y<-rexp(100000)
> z<-rexp(100000)
> x<-x[z<1]
>  y<-y[z<1]
>  z<-z[z<1]
> mean((x<y)*(y<z))
[1] 0.06709503
```

OK, it checks out!

Consider an integral of the form

$$\int \cdots_A \int f(x_1, ..., x_n) dx_1 dx_2 \cdots dx_n,$$

taken over a subset of an n-dimensional space $\mathcal{A}$. Let

$$y_1 = g_1(x_1, x_2, ..., x_n),\ y_2 = g_2(x_1, x_2, ..., x_n), ..., y_n = g_n(x_1, x_2, ..., x_n)$$

define a one-to-one transformation that maps $\mathcal{A}$ onto $\mathcal{B}$.

Denote the corresponding inverse function by

$$x_1 = h_1(y_1, y_2, ..., y_n),\ x_2 = h_2(y_1, y_2, ..., y_n), ..., x_n = h_n(y_1, y_2, ..., y_n)$$

Assume that the partial derivatives of the inverse functions are continuous and not identically equal to 0 in $\mathcal{B}$.

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

Then

$$\int \cdots_A \int f(x_1, ..., x_n) dx_1 dx_2 \cdots dx_n$$

$$= \int \cdots_B \int f[h_1(y_1, ..., y_n), ..., h_n(y_1, ..., y_n)] |J| dy_1 dy_2 \cdots dy_n$$

Example 5: Let $X_1, X_2, ..., X_{k+1}$ be independent random variables, each having a gamma distribution with $\beta = 1$.

$$f(x_1, x_2, ..., x_{k+1}) = \prod_{i=1}^{k+1} \frac{1}{\Gamma(\alpha_i)} x_i^{\alpha_i - 1} e^{-x_i}$$

for $0 < x_i < \infty$.
Let

$$Y_i = \frac{X_i}{X_1 + X_2 + \cdots + X_{k+1}}$$

for $i = 1, 2, ..., k$.

Finally, in order to have a one-to-one transformation, let

$$Y_{k+1} = X_1 + X_2 + \cdots + X_{k+1}$$

.

$\mathcal{A}$=$\{(x_1, x_2, ..., x_{k+1}) : 0 < x_i < \infty\}$

$\mathcal{B}$=$\{(y_1, ..., y_k, y_{k+1}) : 0 < y_i, i = 1, .., k, y_1 + y_2 + \cdots + y_k < 1, 0 < y_{k+1} < \infty\}$.

The inverse functions are

$$x_i = h_i(y_1, ..., y_n) = y_i y_{k+1}$$

for $i = 1, ..., k$

and

$$x_{k+1} = h_{k+1}(y_1, ..., y_n) = y_{k+1}(1 - y_1 - y_2 - \cdots - y_k)$$

We can see that partial derivatives of the inverse functions yield the Jacobian

$$J = \begin{vmatrix} y_{k+1} & 0 & \cdots & 0 & y_1 \\ 0 & y_{k+1} & \cdots & 0 & y_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & y_{k+1} & y_k \\ -y_{k+1} & -y_{k+1} & \cdots & -y_{k+1} & (1 - y_1 - y_2 - \cdots - y_k) \end{vmatrix} = y_{k+1}^k$$

To see that $J = y_{k+1}^k$, one can use the following result about determinants ants:

Let A be a $p \times p$ matrix with i, j element a_{ij}. The **cofactor** of a_{ij} denoted by A_{ij} is $(-1)^{i+j}$ times the determinant of A after deleting the ith row and the jth column.

$$|A| = \sum_{j=1}^{p} a_{ij} A_{ij} = \sum_{i=1}^{p} a_{ij} A_{ij}$$

Consider the case where $k + 1 = 3$.

$$J = \begin{vmatrix} y_3 & 0 & y_1 \\ 0 & y_3 & y_2 \\ -y_3 & -y_3 & (1-y_1-y_2) \end{vmatrix} =$$

Summing along the 3rd row we have

$$J = (-1)^{3+1}(-y_3)\begin{vmatrix} 0 & y_1 \\ y_3 & y_2 \end{vmatrix} + (-1)^{3+2}(-y_3)\begin{vmatrix} y_3 & y_1 \\ 0 & y_2 \end{vmatrix}$$
$$+(-1)^{3+3}(1-y_1-y_2)\begin{vmatrix} y_3 & 0 \\ 0 & y_3 \end{vmatrix}$$

$$= y_3^2 y_1 + y_3^2 y_2 + y_3^2(1-y_1-y_2) = y_3^2$$

Now, recall the density of $X_1, ..., X_n$.

$$f(x_1, x_2, ..., x_{k+1}) = \prod_{i=1}^{k+1} \frac{1}{\Gamma(\alpha_i)} x_i^{\alpha_i - 1} e^{-x_i}$$

If we apply our change of variables formula we find that the joint pdf of $Y_1, Y_2, ..., Y_k, Y_{k+1}$ is

$$f[h_1(y_1, ..., y_{k+1}), ..., h_{k+1}(y_1, ..., y_{k+1})]|J|$$

$$= \frac{y_{k+1}^{\alpha-1} y_1^{\alpha_1-1} \cdots y_k^{\alpha_k-1}(1 - y_1 - \cdots - y_k)^{\alpha_{k+1}-1} e^{-y_{k+1}}}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_k)\Gamma(\alpha_{k+1})}$$

where $\alpha = \Sigma_{i=1}^{k+1} \alpha_i$.

We see that terms involving y_{k+1} in this pdf enter multiplicatively, so that Y_{k+1} is independent of $Y_1, Y_2, ..., Y_k$.

If we integrate this pdf over the space of Y_{k+1} Y_{k+1} we obtain the pdf of $Y_1, Y_2, ..., Y_k$.

$$f(y_1, y_2, ..., y_n) =$$

$$\frac{\Gamma(\alpha)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_{k+1})} y_1^{\alpha_1 - 1} \cdots y_k^{\alpha_k - 1}(1 - y_1 - \cdots - y_k)^{\alpha_{k+1}-1}$$

when $0 < y_i$ and $\Sigma_{i=1}^{k} y_i < 1$.

Random variables $Y_1, ..., Y_k$ with a joint pdf of this form are said to have a **Dirichlet distribution** with parameters $\alpha_1, ..., \alpha_k, \alpha_{k+1}$.

Also, note that Y_{k+1} has a gamma distribution with parameters $\alpha = \Sigma_{i=1}^{k+1} \alpha_i$ and $\beta = 1$.

Example: Let's try another one!